General Exam Solutions
Fall 1998

Problem 1 (Mechanics)

A uniform string of length L is tied at its two ends along the x axis and stretches from $x = 0$ to $x = L$. The mass per unit length of the string is ρ. For times $t < 0$ the tension is a constant, T_0, and the string is given to be oscillating in the form

$$y(x,t) = A\sin(\omega t)\sin\left(\frac{\pi x}{L}\right), \quad t < 0, \qquad (1)$$

where y is the transverse displacement of the string.

a) Write the Lagrangian L for a string of mass per unit length ρ and tension T_0. Note that in addition to kinetic energy from the motion, there is potential energy from the stretching of the string. From this Lagrangian find the Lagrangian equation of motion of the string. ✓

b) Using Newton's second law applied to an infinitesimal piece of string rederive the equation of motion obtained in part (a). ✓

c) What is the value of the frequency ω in the expression (1) for $y(x,t)$? ✓

We now consider the possibility that the tension is time dependent. For this purpose one may simply replace T_0 by $T(t)$ in the Lagrangian and in the equation of motion. In the time interval $0 < t < t_0$ the tension is increased to a slightly higher value $T_0(1+\epsilon)$, with ϵ a small constant, after which the tension returns to the value T_0.

d) Assuming a simple 'separation of variables' form $y(x,t) = X(x)Y(t)$ write out the equations for the functions $X(x)$, $Y(t)$. Solve for $X(x)$. ✓

e) Rewrite the equation for $Y(t)$ in the form $\frac{d^2Y}{dt^2} + \omega^2 Y = \cdots$, where the dots represent the perturbation term linear in ϵ. Using first order perturbation theory in the small constant ϵ, solve for $Y(t)$ for times $t > t_0$.

Useful information The Green's function for the equation $\frac{d^2y}{dt^2} + \omega^2 y = 0$ is

$$\begin{aligned} G(t,t') &= \frac{1}{\omega}\sin\left[\omega(t-t')\right], \quad \text{for } t > t' \\ &= 0, \quad \text{for } t < t'. \end{aligned}$$

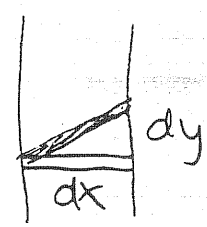
It satisfies $\frac{d^2G(t,t')}{dt^2} + \omega^2 G(t,t') = \delta(t-t')$.

Problem 1 Mechanics

a) $L = T - V$

$$T = \int_0^L \frac{1}{2} \underbrace{(\rho dx)}_{dm} \dot{y}^2 = \frac{1}{2}\int_0^L dx\, \rho \dot{y}^2$$

$$V = \int_0^L T_0 \underbrace{\frac{1}{2} dx (y')^2}_{\text{stretch}}$$



$\frac{dy}{dx} \ll 1$

stretch:

$$\sqrt{dx^2 + dy^2} - dx$$

$$= dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} - dx$$

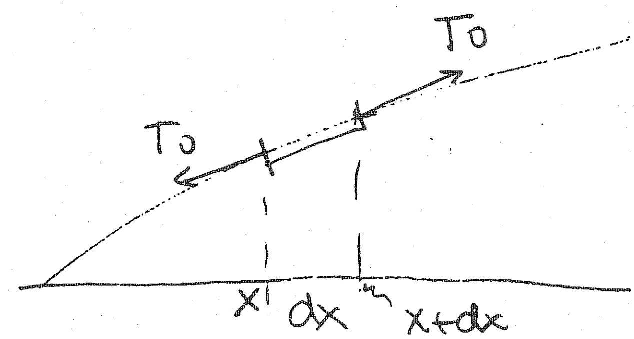
$$\simeq \frac{1}{2} dx (y')^2$$

$$\boxed{L = \frac{1}{2}\int_0^L dx \left(\rho \dot{y}^2 - T_0 y'^2\right)}$$

EOMotion

$$\rho \ddot{y} = T_0 y''$$

b)



Force upwards $\quad T_0 \left.\dfrac{dy}{dx}\right|_{x+dx} - T_0 \left.\dfrac{dy}{dx}\right|_{x}$

$$= T_0 \frac{d^2y}{dx^2}\, dx = (\rho\, dx)\, \frac{d^2y}{dt^2}$$

$$(dF) = (dm)\, a$$

$$\rightarrow \quad T_0\, y'' = \rho\, \ddot{y} \quad \cdots (\alpha)$$

c) Plug (1) into (α)

$$T_0 \left(\frac{\pi}{L}\right)^2 = \rho\, \omega^2 \rightarrow \boxed{\omega = \sqrt{\frac{T_0}{\rho}}\, \frac{\pi}{L}}$$

d)

$$T(t)\, X''(x)\, Y(t) = \rho\, X(x)\, Y''(t)$$

$$\frac{X''(x)}{X(x)} = \frac{\rho}{T(t)} \frac{Y''(t)}{Y(t)} = -\alpha^2$$

constant

$$X''(x) + \alpha^2 X(x) = 0$$

since $X(0) = X(L) = 0$, we keep the same solution, cannot jump mode

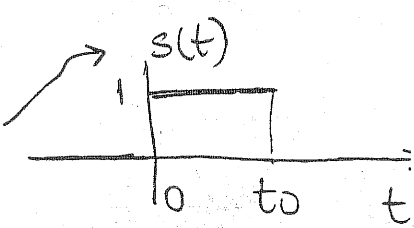
$$\alpha^2 = \left(\frac{\pi}{L}\right)^2 \qquad \boxed{X(x) = A \sin\left(\frac{\pi x}{L}\right)}$$

this consta fits (1) for convenien[illegible]

$$Y''(t) + \alpha^2 \frac{T(t)}{\rho} Y(t) = 0$$

$$Y''(t) + \underbrace{\left(\frac{\pi}{L}\right)^2 \frac{T_0}{\rho}}_{\omega^2} \left(1 + \epsilon\, s(t)\right) Y(t) = 0$$

here



$$\boxed{Y''(t) + \omega^2 Y(t) = -\omega^2 \epsilon\, s(t)\, Y(t)}$$

e) $Y(t) = \sin \omega t$ for $t<0$ to match

in perturb theory:

$$Y(t) = \sin \omega t + \int_{-\infty}^{\infty} dt'\, G(t,t') \left(-\omega^2 \epsilon\, s(t') Y\right.$$

for $t > t_0$ since $G(t,t') = 0$ for $t < t'$

$$Y(t) = \sin \omega t + \int_{-\infty}^{t_0} dt'\, G(t,t') \left(-\omega^2 \epsilon\, s(t') Y(t\right.$$

since $S(t')$ vanishes for $t'<0$

$$Y(t) = \sin\omega t - \omega^2\epsilon \int_0^{t_0} dt'\, G(t,t')\, (\underline{\underline{\sin\omega t'}})$$

this is the zeroth order Y

$$\boxed{Y(t) = \sin\omega t - \epsilon\omega \int_0^{t_0} dt' \sin\omega(t-t') \sin\omega t'}$$

$$\sin A \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$

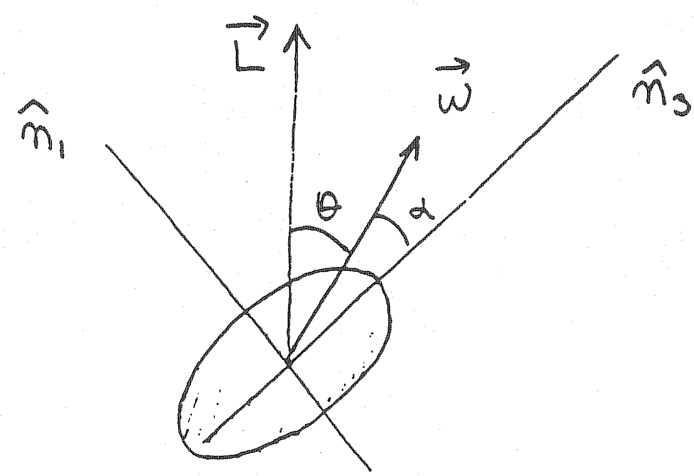
$$\sin(\omega t - \omega t') \sin\omega t' = \frac{1}{2}\cos(\omega t - 2\omega t') - \frac{1}{2}\cos\omega t$$

$$Y(t) = \sin\omega t - \epsilon\omega \frac{1}{2}\int_0^{t_0} dt' \cos(\omega t - 2\omega t') + \epsilon\omega\frac{1}{2}\int_0^{t_0} dt' \cos\omega t$$

$$\boxed{Y(t) = \sin\omega t + \frac{1}{2}\epsilon(\omega t_0)\cos\omega t - \frac{1}{4}\epsilon\sin\omega t - \frac{1}{4}\epsilon\sin(2\omega t_0 - \omega t)}$$

Problem 2 (Mechanics)

Consider a prolate spheroid with no external forces acting on it. At $t = 0$ the spheroid is given an angular velocity of magnitude ω about its center of mass, in a direction which is inclined at angle α to the axis of symmetry.



Use 'body axes' such that the unit vector $\hat{n}_3$ points along the symmetry axis of the spheroid, and the unit vectors $\hat{n}_1$ and $\hat{n}_2$ lie in the plane perpendicular to this axis. The moments of inertia are $I_1 = I_2 = I$ and $I_3 = (1 - a)I$, where $0 < a < 1$ is a constant.

a) Write the angular velocity vector $\vec{\omega}$ at $t = 0$ using the unit vectors $\hat{n}_1$, $\hat{n}_2$ and $\hat{n}_3$. Assume that this vector lies in the $\hat{n}_1$ - $\hat{n}_3$ plane at $t = 0$.

b) Find the angular momentum vector $\vec{L}$ at $t = 0$ in this basis.

The *space cone* is generated by the motion of $\vec{\omega}$ around $\vec{L}$, and the *body cone* is generated by the motion of $\vec{\omega}$ around the symmetry axis of the body. The half angle of the body cone is α and the half angle of the space cone is θ (see figure).

c) Find the half angle θ of the space cone in terms of α and a.

d) Find the angular frequency Ω_b for the rotation of $\vec{\omega}$ around the axis of symmetry of the spheroid (as seen by an observer fixed on the spheroid, $\vec{\omega}$ spins around the symmetry axis).

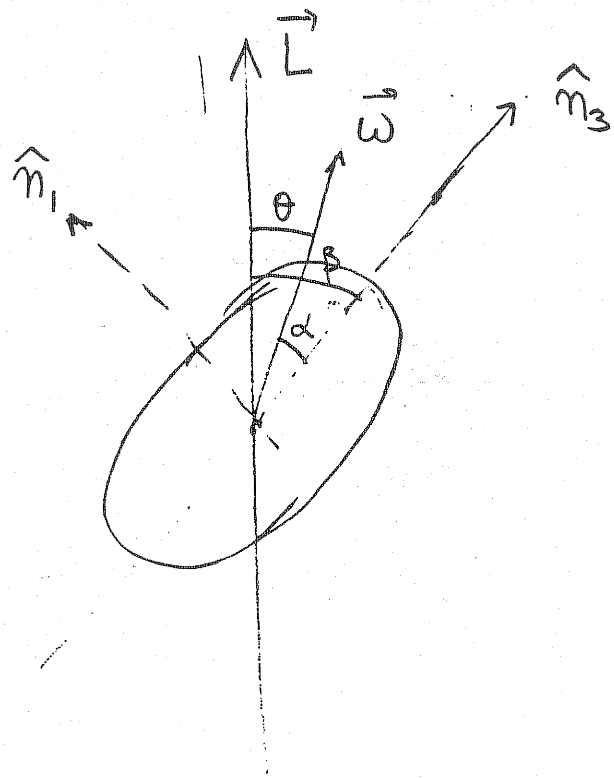
e) Find the angular frequency Ω_p for the rotation of $\vec{\omega}$ around $\vec{L}$. Give your answer in terms of I and the magnitude L of $\vec{L}$.

Useful information:

$$\left(\frac{d\vec{L}}{dt}\right)_s = \left(\frac{d\vec{L}}{dt}\right)_b + \vec{\omega} \times \vec{L}$$

where 's' denotes space fixed coordinates, and 'b' denotes body fixed coordinates.

Problem 2 Solution



$I_1 = I_2 = I$

$I_3 = (1-a) I$

$$\mathbb{I} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1-a \end{pmatrix} I$$

a) $$\vec{\omega} = \begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix} \omega \quad \begin{array}{l} \leftarrow 1 \\ \leftarrow 2 \\ \leftarrow 3 \end{array} \quad \text{or} \quad \vec{\omega} = \omega(\sin\alpha\, \hat{n}_1 + \cos\alpha\, \hat{n}_3)$$

b) $$\vec{L} = I\omega = I\omega \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1-a \end{pmatrix} \begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix}$$

$$\vec{L} = I\omega \begin{pmatrix} \sin\alpha \\ 0 \\ (1-a)\cos\alpha \end{pmatrix}$$

c) let β be the angle from L to $\hat{n}_3$

$$\tan\beta = \frac{\sin\alpha}{\cos\alpha} \frac{1}{(1-a)} \qquad \tan\beta = \frac{1}{(1-a)} \cdot \tan\alpha$$

$$\boxed{\theta = \beta - \alpha = \tan^{-1}\left[\frac{1}{1-a}\tan\alpha\right] - \alpha}$$

θ : half angle for <u>space cone</u>

d) $$\frac{dL}{dt} + \omega \times L = 0 \qquad (\vec{N} = 0, \text{ use body axes})$$

$$I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = 0$$

$$I_2\dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3) = 0$$

$$I_3\dot{\omega}_3 + \omega_1\omega_2\underbrace{\cancel{(I_2 - I_1)}}_{0} = 0$$

$$\dot{\omega}_3 = 0 \qquad \omega_3 \text{ is a constant}$$

$$\boxed{\omega_3 = \omega\cos\alpha}$$

$$I_3 - I_2 = [(1-a) - 1]\, I = -aI$$

$$I_1 - I_3 = [1 - (1-a)]\, I = aI$$

$$\rightarrow \quad I\dot{\omega}_1 + \omega_2\omega_3(-aI) = 0$$

$$I\dot{\omega}_2 + \omega_3\omega_1(aI) = 0$$

$$\dot{\omega}_1 = (a\omega_3)\,\omega_2$$

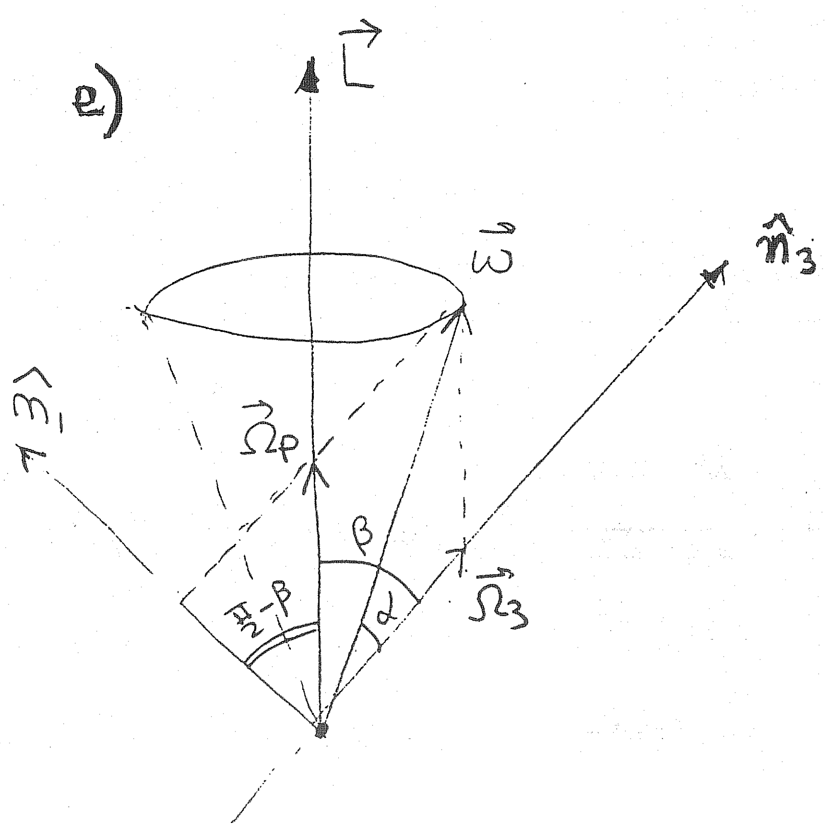
$$\dot{\omega}_2 = -(a\omega_3)\,\omega_1$$

$$\rightarrow \quad \ddot{\omega}_1 = -(a\omega_3)^2\,\omega_1$$

the angular frequency here is

$$\boxed{\Omega_b = a\omega_3 = a\omega\cos\alpha}$$

e)



$\hat{n}_3$, $\vec{\omega}$ and $\vec{L}$ form a plane at any time

$\vec{L}$ is fixed in space

$\vec{\omega}$ precesses around $\vec{L}$

$\vec{\omega}$ is decomposed (uniquely) as the sum of a vector $\vec{\Omega}_p$ along $\vec{L}$ and a vector $\vec{\Omega}_3$ along $\hat{n}_3$

$|\vec{\Omega}_p|$ is the desired angular velocity

$$\Omega_p \cos\left(\frac{\pi}{2}-\beta\right) = \omega_1$$

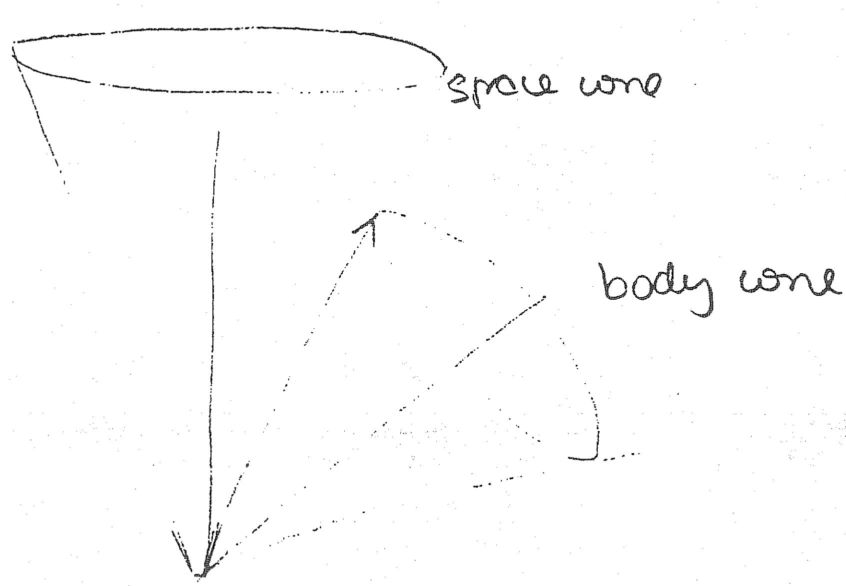
$$\Omega_p \sin\beta = \omega_1$$

$$\Omega_p = \frac{\omega_1}{\sin\beta} = \frac{\sin\alpha}{\sin\beta}\,\omega$$

$$\boxed{\Omega_p = \frac{\sin\alpha}{\sin(\alpha+\theta)}\,\omega}$$

$$\sin\beta = \frac{L_1'}{L} = \frac{I\omega_1}{L}$$

$$\boxed{\Omega_p = \frac{\omega_1}{\frac{I\omega_1}{L}} = \frac{L}{I}}$$



Problem 1 (Electromagnetism)

An infinitely long wire on the z-axis has no current for $t < 0$ and a constant current I for $t > 0$ (in the positive z-direction). As in an ordinary wire, where the fixed positive ions cancel the charge of the moving electrons, there is no charge density for all t.

a) Express the current density $\vec{J}(\vec{r}, t)$ using δ-functions and/or step functions.

b) What is the vector potential $\vec{A}(\vec{r}, t)$ for this current? Use the retarded Green's functions and evaluate all integrals. Hints: $\vec{A}(\vec{r}, t) = \vec{A}(\rho, t)$ where ρ is the normal distance to the wire. As the answer is independent of z, consider the observer at $z = 0$.

c) Find the magnetic field $\vec{B}(\vec{r}, t)$ corresponding to $\vec{A}(\vec{r}, t)$. Plot your answer in two ways; in the first show the magnetic field at a fixed point as a function of time, and in the second show the magnetic field at a fixed time as a function of the radial distance to the wire.

d) Find the electric field $\vec{E}(\vec{r}, t)$. What is its behavior at large times?

Useful Formulas:

$$\vec{A}(\vec{r}, t) = \frac{1}{c}\int d^3r' \, \frac{\vec{J}\left[\vec{r}\,', t' = t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right]}{|\vec{r}-\vec{r}\,'|}$$

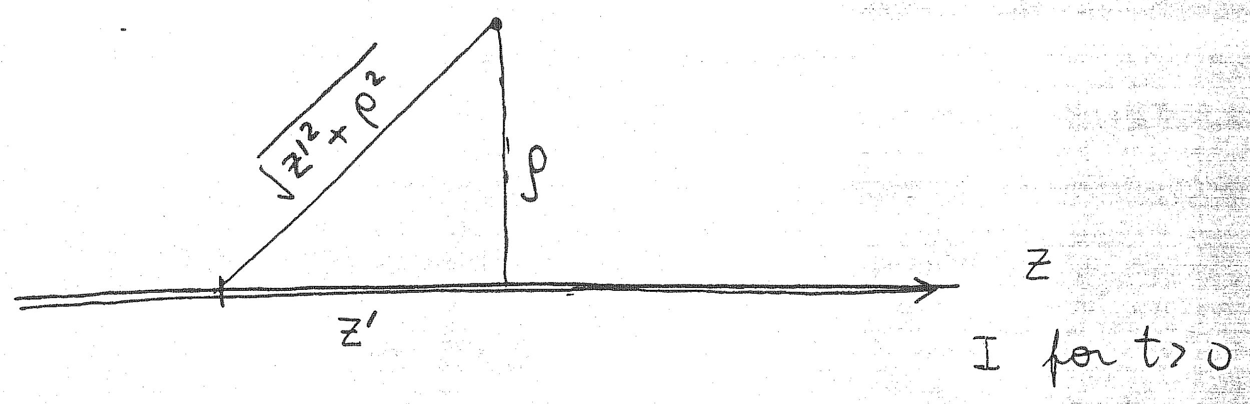
$$\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u$$

In cylindrical coordinates

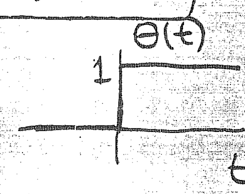
$$\nabla \times \vec{a} = \begin{vmatrix} \hat{e}_\rho & \hat{e}_\varphi & \hat{e}_z \\ \frac{\partial}{\partial \rho} & \frac{1}{\rho}\frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ a_\rho & a_\varphi & a_z \end{vmatrix}$$

Electromagnetism Problem 1



a) $\vec{J}(\vec{r},t) \begin{cases} \hat{z}\, I\, \delta(x)\,\delta(y) & t>0 \\ 0 & t<0 \end{cases}$

or write $\boxed{\vec{J}(\vec{r},t) = \hat{z}\, I\, \delta(x)\,\delta(y)\, \Theta(t)}$



b) $\vec{A}(\vec{r},t) = \frac{1}{c}\int dx'dy'dz' \, \frac{\vec{J}(\vec{r}',t')}{|\vec{r}-\vec{r}'|}$

only z-component, current at $x=y=0$

$$A_z(\rho,t) = \frac{I}{c}\int dz' \, \frac{\Theta(t')}{\sqrt{z'^2+\rho^2}}$$

with $t' = t - \frac{\sqrt{z'^2+\rho^2}}{c}$

For a fixed ρ consider also a fixed time t

If $\boxed{t < \frac{\rho}{c} \qquad A_z(\rho,t) = 0}$ since

$t' < t - \frac{\rho}{c} < 0$

and at $t' < 0$ there is no current on the wire

$\underline{\underline{t > \frac{\rho}{c}}}$; we must integrate on the wire from $-z'$ to z'

where $\sqrt{z'^2 + \rho^2} = ct$ (this makes t' positive)

$$A_z(\rho,t) = \frac{I}{c}\int_{-z'}^{z'} dz \frac{1}{\sqrt{\rho^2+z^2}} \qquad z = \rho x$$

$$= \frac{I}{c}\int_{-z'/\rho}^{z'/\rho} dx \frac{1}{\sqrt{1+x^2}}$$

$$= 2\frac{I}{c} \sinh^{-1} \frac{z'}{\rho}$$

$$\boxed{A_z(\rho,t) = \frac{2I}{c}\sinh^{-1}\left(\frac{\sqrt{c^2t^2-\rho^2}}{\rho}\right)} \qquad t > \frac{\rho}{c}$$

c) Find the magnetic field

$$\vec{B} = \vec{\nabla} \times \vec{A} = -\vec{e}_\phi \frac{\partial A_z}{\partial \rho}$$

$$B_\phi = -\frac{\partial A_z}{\partial \rho} = -\frac{\partial}{\partial \rho}\left[\frac{2I}{c}\sinh^{-1}\left(\frac{z'}{\rho}\right)\right]$$

$$= -\frac{2I}{c}\frac{1}{\sqrt{1+\frac{z'^2}{\rho^2}}}\frac{\partial}{\partial \rho}\left(\frac{z'}{\rho}\right)$$

$$= -\frac{2I}{c}\left(\frac{\rho}{ct}\right)\left[\frac{1}{\rho}\frac{\partial z'}{\partial \rho} - \frac{z'}{\rho^2}\right]$$

from $\quad z'^2 + \rho^2 = c^2t^2$

$$2z'\frac{\partial z'}{\partial \rho} + 2\rho = 0$$

$$z'\frac{\partial z'}{\partial \rho} = -\rho \qquad \frac{1}{\rho}\frac{\partial z'}{\partial \rho} = -\frac{1}{z'}$$

$$B_\phi = -\frac{2I}{c}\left(\frac{\rho}{ct}\right)\left[-\frac{1}{z'} - \frac{z'}{\rho^2}\right]$$

$$= \frac{2I}{c}\left(\frac{\rho}{ct}\right)\frac{(\rho^2+z'^2)}{z'\rho^2} = \left(\frac{2I}{c}\right)\frac{\cancel{\rho}}{ct}\frac{c^2t^2}{z'\rho^{\cancel{2}}}$$

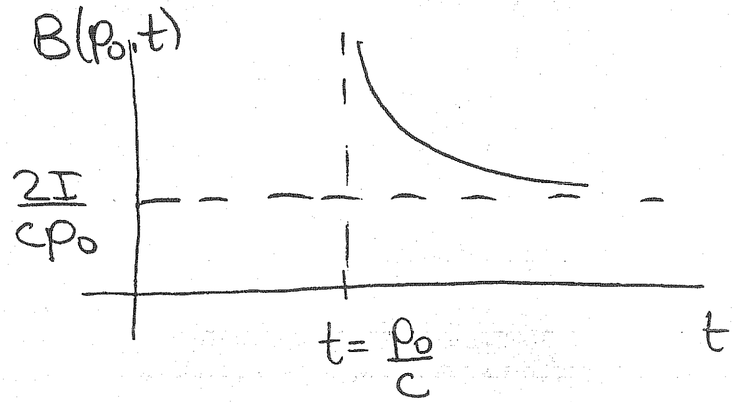
$$= \frac{2I}{c\rho}\frac{ct}{z'}$$

$$B_\phi = \frac{2I}{c\rho}\frac{ct}{\sqrt{c^2t^2-\rho^2}}$$

$$\boxed{B_\phi(\rho,t) = \frac{2I}{c\rho}\frac{1}{\sqrt{1-\left(\frac{\rho}{ct}\right)^2}}} \qquad t > \frac{\rho}{c}$$

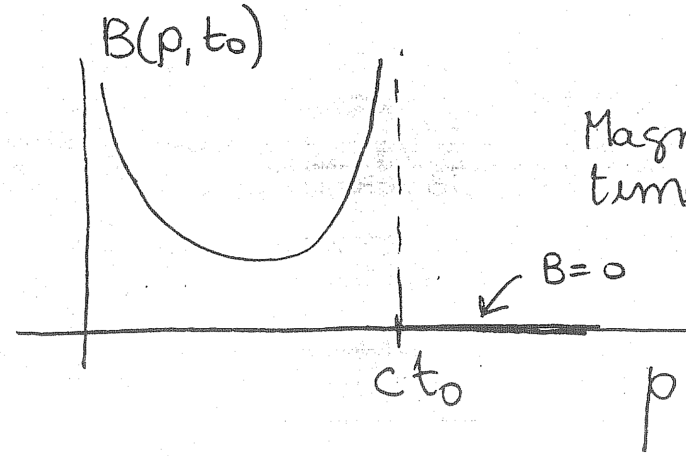
$B_\phi = 0$ otherwise

Magnetic field at a fixed point as a function of time $B_\phi(\rho_0, t)$



$t_0 > 0$

Magnetic field at fixed time as a function of distance.



d) Electric field :

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$\Phi = 0$ since there is no charge density

only z-component

$$E_z = -\frac{1}{c}\frac{\partial A_z}{\partial t}$$

$$= -\frac{1}{c}\left(\frac{2I}{c}\right)\frac{\partial}{\partial t}\sinh^{-1}\left(\frac{z'}{\rho}\right)$$

$$= -\frac{2I}{c^2}\left(\frac{\rho}{ct}\right)\frac{\partial}{\partial t}\left(\frac{z'}{\rho}\right)$$

$$= -\frac{2I}{c^2}\frac{1}{ct}\frac{\partial z'}{\partial t} \quad ; \quad 2z'\frac{\partial z'}{\partial t} = 2c^2 t$$

$$= -\frac{2I}{\cancel{c^2}}\frac{1}{c\cancel{t}}\frac{\cancel{c^2}\cancel{t}}{z'} \qquad \frac{\partial z'}{\partial t} = \frac{c^2 t}{z'}$$

$$\boxed{E_z = -\frac{2I}{c}\frac{1}{\sqrt{c^2t^2-\rho^2}}} \qquad t > \frac{\rho}{c}$$

$\vec{E} = E_z\hat{z}$ $\qquad \vec{E} = 0$ for $t < \frac{\rho}{c}$

for large t

$$\boxed{E_z \approx -\frac{2I}{c^2 t} \longrightarrow 0}$$

Problem 2 **(Electromagnetism)**

A plane wave travelling along the positive z-direction is incident normally on a uniform material filling the half-space $z \geq 0$. The material has a constant conductivity $\sigma > 0$, and $\epsilon = \mu = 1$.

The incident field (for $z < 0$) is of the form

$$\vec{E}_{\text{inc}}(\vec{r}, t) = Re\left\{E_0\, e^{i(kz - \omega t)}\, \hat{e}_x\right\}, \tag{1}$$

with $k = \frac{\omega}{c}$ and E_0 a real constant. Consider the simple *ansatz* for the wave in the conductor ($z > 0$)

$$\vec{E}_{\text{con}}(\vec{r}, t) = Re\left\{\vec{E}_c(\vec{r})\, e^{-i\omega t}\right\}. \tag{2}$$

Here, the possibly complex vector $\vec{E}_c(\vec{r})$ is to be determined. For $z < 0$ we assume that in addition to the incident wave there is a reflected wave of the form

$$\vec{E}_{\text{ref}}(\vec{r}, t) = Re\left\{E_r\, e^{i(-kz - \omega t)}\, \vec{e}_x\right\}, \tag{3}$$

propagating in the $(-z)$ direction. Here E_r is a constant to be found.

a) Consider Maxwell's equations for the time-independent fields $\vec{E}_c(\vec{r})$, $\vec{B}_c(\vec{r})$ in the conducting media. Use $\vec{J}_c = \sigma \vec{E}_c$, and show that the differential equation for $\vec{E}_c(\vec{r})$. takes the form

$$\left\{\nabla^2 + k^2\left(1 + i\frac{4\pi\sigma}{\omega}\right)\right\} \vec{E}_c(\vec{r}) = 0\ . \tag{4}$$

b) Assume now that the fields in the conductor are of the form

$$\begin{aligned} \vec{E}_c(\vec{r}) &= E_c\, e^{i\beta z}\, \hat{e}_x\,, \\ \vec{B}_c(\vec{r}) &= B_c\, e^{i\beta z}\, \hat{e}_y\,. \end{aligned} \tag{5}$$

What is the value of β in terms of k, σ, ω ? Find B_c in terms of E_c.

c) For the reflected wave find the time-independent magnetic field in terms of the constant E_r introduced in (3).

d) What are the relevant boundary conditions for $\vec{E}$ *and* $\vec{B}$ at the boundary ? Set up and solve the system of equations that determine E_c and E_r in terms of E_0, k and β.

e) Using the stress-tensor T_{ij} find the pressure on the conducting wall due to the radiation. (Draw a sketch showing the surface you will use) Express your answer in terms of E_c, k and β.

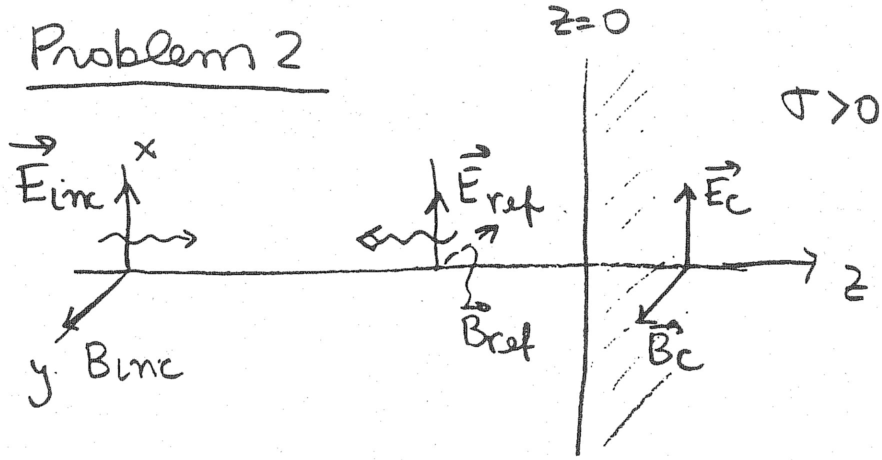
Useful Information:

$$\nabla \times (\nabla \times \vec{a}) = \nabla(\nabla \cdot \vec{a}) - \nabla^2 \vec{a}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$T_{ij} = \frac{1}{4\pi}\left[E_iE_j + B_iB_j - \frac{1}{2}\delta_{ij}\left(|\vec{E}|^2 + |\vec{B}|^2\right)\right]$$

Problem 2



$\sigma > 0 \qquad (\epsilon = \mu = 1)$

$k = \frac{\omega}{c}$

$$\vec{E}_{inc}(\vec{r},t) = Re\left\{ E_0 e^{i(kz-\omega t)} \hat{e}_x \right\}$$

$$\vec{E}_{cond}(r,t) = Re\left\{ \vec{E}_c(\vec{r})\, e^{-i\omega t} \right\}$$

$$\vec{E}_{ref}(r,t) = Re\left\{ E_r e^{i(-kz-\omega t)} \vec{e}_x \right\}$$

a) $\nabla \times \vec{E}(t) = -\frac{1}{c}\frac{\partial \vec{B}(t)}{\partial t}$; $\qquad e^{-i\omega t}$

$$\longrightarrow \boxed{\nabla \times \vec{E}_c(\vec{r}) = ik\vec{B}_c(\vec{r})} \qquad \nabla \times \vec{B}_c(t) = \frac{4\pi \vec{J}_c(t)}{c} + \frac{1}{c}\frac{\partial E_c(t)}{\partial t}$$

$$\nabla \times \vec{B}_c = \frac{4\pi\sigma \vec{E}_c}{c} - ik\vec{E}_c$$

$$\nabla \times \vec{B}(r) = -ik\left(1 + i\frac{4\pi\sigma}{kc}\right)\vec{E}(\vec{r})$$

$$\boxed{\nabla \times B_c(\vec{r}) = -ik\left(1 + i\frac{4\pi\sigma}{\omega}\right)\vec{E}_c(\vec{r})}$$

div curl = 0

$\to$ div E = 0

$$\nabla \times (\nabla \times \vec{E}_c) = ik\, \nabla \times \vec{B}_c$$

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}_c = (ik)(-ik)\left(1 + i\frac{4\pi\sigma}{\omega}\right)\vec{E}_c$$

$$-\nabla^2 E_c = k^2\left(1 + i\frac{4\pi\sigma}{\omega}\right)\vec{E}_c$$

$$\boxed{\left[\nabla^2 + k^2\left(1 + i\frac{4\pi\sigma}{\omega}\right)\right] E_c(\vec{r}) = 0} \quad \cdots (\mathrm{I})$$

b) $$\vec{E}_c(\vec{r}) = E_c\, e^{i\tilde{k}z}\, \hat{e}_x$$

$$\vec{B}_c(r) = B_c\, e^{i\tilde{k}z}\, \hat{e}_y$$

from(I) $\longrightarrow$ $$\tilde{k}^2 = k^2\left(1 + i\frac{4\pi\sigma}{\omega}\right)$$

$$\boxed{\tilde{k} = k\left(1 + i\frac{4\pi\sigma}{\omega}\right)^{1/2}}$$

Use $\nabla \times \vec{E}_c = ik\vec{B}_c(r)$

$$\nabla \times E_c = \begin{vmatrix} \overset{+}{e_x} & \overset{-}{e_y} & \overset{+}{e_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_c e^{i\tilde{k}z} & 0 & 0 \end{vmatrix}$$

$$= + \frac{\partial}{\partial z} E_c e^{i\tilde{k}z}\, \vec{e}_y$$

$$= i\tilde{k}\, E_c\, \hat{e}_y\, e^{i\tilde{k}z}$$

$\rightarrow$ $i\tilde{k} E_c = ik B_c$

$$\boxed{B_c = \frac{\tilde{k} E_c}{k}}$$

c) The time indep. reflected field $\vec{E}$ is

$$\vec{E}_{ref}(\vec{r}) = E_r\, e^{-ikz}\, \vec{e}_x$$

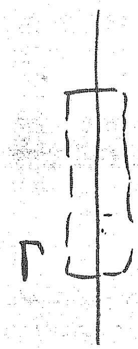
and $$B_{ref}(\vec{r}) = B_r\, e^{-ikz}\, \vec{e}_y$$

$$\nabla \times \vec{E} = ik\vec{B} \longrightarrow -E_r = B_r \qquad \boxed{B_r = -E_r}$$

d) Boundary conditions

E_{tans} is continuous

B_{tans} is continuous



since $\nabla \times \bar{E} \sim B$

$\nabla \times B \sim E$

$\oint_{\Gamma} \vec{E} \cdot d\vec{l} \sim \int B \cdot dA \rightarrow 0$

$\oint \vec{B} \cdot \vec{dl} \sim \int E \cdot dA \rightarrow 0$

at $z = 0$

E_{tan} continuous:

$$E_0 + E_r = E_c \quad \text{....} (\alpha)$$

B_{tan} continuous:

$$E_0 - E_r = \frac{\tilde{k}}{k} E_c \quad \text{...} (\beta)$$

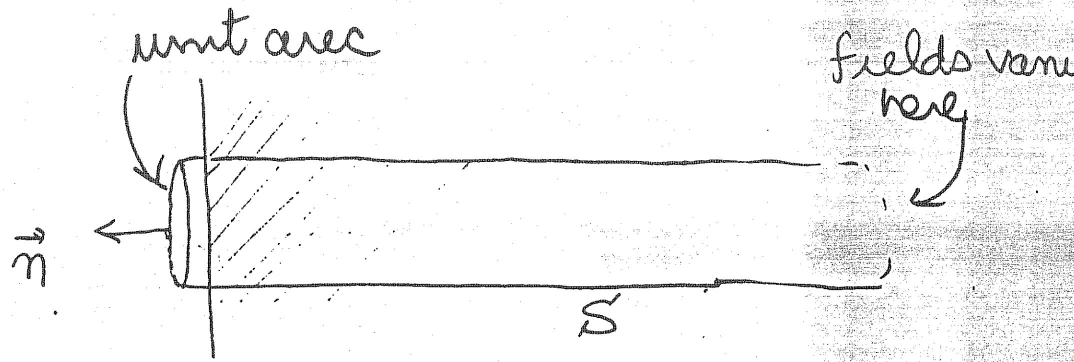
Add:

$$2E_0 = \left(1 + \frac{\tilde{k}}{k}\right) E_c \qquad \boxed{E_c = \frac{2k}{k + \tilde{k}} E_0}$$

$$E_r = E_c - E_0 = \left(\frac{2k}{k + \tilde{k}} - 1\right) E_0 = \frac{k - \tilde{k}}{k + \tilde{k}} E_0$$

$$\boxed{E_r = \frac{k - \tilde{k}}{k + \tilde{k}} E_0}$$

e)



$$\int_S T_{\alpha\beta}\, n_\beta \, ds$$ is the α- component of force on the body

S ↑ surface; n_β ↑ outgoing normal of S

$n = (0, 0, -1)$

want z- component

$$\text{Pressure on object} = T_{3\beta}\, n_\beta = -T_{33}$$

$$T_{33} = \frac{1}{4\pi}\left[\underbrace{\cancel{E_3 E_3}}_{0} + \underbrace{\cancel{B_3 B_3}}_{0} - \frac{1}{2}\left(|E|^2 + |B|^2\right)\right]$$

$$\text{pressure} = \frac{1}{8\pi}\left(|E|^2 + |B|^2\right)$$ evaluated an ϵ to the left of the wall

$$E = E_0 + E_r = E_c$$
$$B = E_0 - E_r = \frac{\tilde{k}}{k} E_c$$

$$\underline{\text{pressure}} = \frac{1}{8\pi}\left(E_c E_c^* + \frac{\tilde{k}\tilde{k}^*}{k k} E_c E_c^*\right) = \underline{\underline{\frac{1}{8\pi}\left(1 + \frac{|\tilde{k}|^2}{k^2}\right)|E}}$$

Statistical Mechanics: Problem #1

1. The *Dieterici equation of state* relates the pressure P of an interacting gas to its temperature T, and specific volume $v = V/N$, by

$$P = \frac{k_B T}{(v-b)} \exp\left[-\frac{a}{k_B T v}\right]$$

where a and b are positive constants.

(a) Sketch typical isotherms $P(v)$ predicted by the above equation for high and low temperatures. What is wrong with this equation? Sketch the isotherms of a real gas at low tempertures.

(b) Find the coordinates (P_c, v_c, T_c) of the *critical point* of this gas.

(c) Find the singular temperature dependence of the isothermal susceptibility

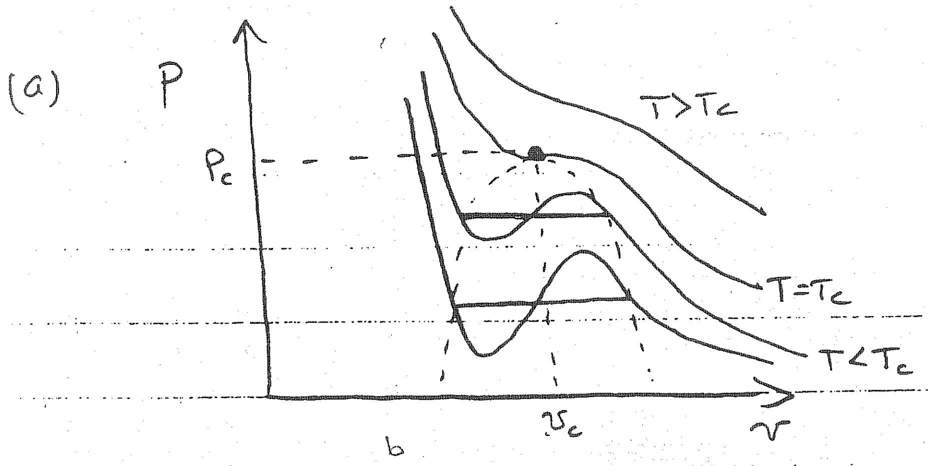
$$K_T \equiv -\frac{1}{v} \frac{\partial v}{\partial P}\bigg|_{T,N} .$$

on approaching the critical point $T \to T_c$, at $v = v_c$.

(d) Find the pressure $P(v, T_c)$ on the critical isotherm $T = T_c$ to lowest non-vanishing order in $(v - v_c)$.

Solution to S.M. #1.

(a)



At low T, the equation results in isotherms that do not correspond to $\frac{\partial P}{\partial v} \leq 0$. This is unphysical, corresponding to negative compressibility and mechanical instability.

In a real gas, in the thermodynamic limit, there is coexistance (i.e. the onset of a phase transition) between liquid and vapor phases, and the isotherms have horizontal coexistence regions (red)

The critical point is the point of inflection $\left.\frac{\partial P}{\partial v}\right|_{T_c,N} = 0$, $\left.\frac{\partial^2 P}{\partial v^2}\right|_{T_c,N} = 0$

$$\left.\frac{\partial P}{\partial v}\right|_{T,N} = \frac{\partial}{\partial v}\left[\frac{k_B T}{(v-b)} \exp\left(-\frac{a}{k_B T v}\right)\right] = P\left[\frac{a}{k_B T v^2} - \frac{1}{(v-b)}\right]$$ and

$$\left.\frac{\partial^2 P}{\partial v^2}\right|_{T,N} = \frac{\partial P}{\partial v}\left[\frac{a}{k_B T v^2} - \frac{1}{(v-b)}\right] - P\left[\frac{2a}{k_B T v^3} - \frac{1}{(v-b)^2}\right].$$ Setting both to zero gives

$$\frac{a}{k_B T_c v_c^2} - \frac{1}{v_c - b} = 0 \quad \text{and} \quad \frac{2a}{k_B T_c v_c^3} - \frac{1}{(v_c-b)^2} = 0$$

with solutions $\boxed{v_c = 2b}$ and $\boxed{k_B T_c = a/4b}$. The critical pressure is then:

$$P_c = \frac{k_B T_c}{v_c - b} \exp\left(-a/k_B T_c v_c\right) = \boxed{\frac{a}{4b^2} e^{-2} = P_c}$$

(c) From (b) we have $\left.\frac{\partial P}{\partial v}\right|_{T,N} = P\left(\frac{a}{k_BTv^2} - \frac{1}{(v-b)}\right)$. Expanding at $v=v_c$, in terms of $\tau \equiv k_BT - k_BT_c$ (for $T>T_c$) yields

$$\left.\frac{\partial P}{\partial v}\right|_{T,N} \simeq P_c\left[\frac{a}{(a/4b+\tau)\cdot 4b^2} - \frac{1}{b}\right] \simeq -\frac{P_c}{b}\cdot\frac{4b\tau}{a} = -\frac{2P_c}{v_c k_BT_c}\tau$$

Thus the compressibility $K_T = -\frac{1}{v}\left.\frac{\partial v}{\partial P}\right|_{T,N}$ diverges as

$$\boxed{K_T = \frac{k_BT_c}{2P_c}\cdot\frac{1}{k_B(T-T_c)} = \frac{T_c}{2P_c}\cdot\frac{1}{(T-T_c)} = \frac{be^2}{2k_B(T-T_c)}}$$

on approaching the critical temperature from above. It is infinite on the coexistence line for $T<T_c$.

(d) at $T=T_c$ we expand around $v=v_c$.

$$P(v,T_c) = P_c + \cancel{\left.\frac{\partial P}{\partial v}\right|_{T_c,v_c}(v-v_c)}^{\,0} + \cancel{\frac{1}{2}\left.\frac{\partial^2 P}{\partial v^2}\right|_{T_c,v_c}(v-v_c)^2}^{\,0} + \frac{1}{6}\left.\frac{\partial^3 P}{\partial v^3}\right|_{T_c,v_c}(v-v_c)^3 + \dots$$

because of critical point.

$$P(v,T_c) = P_c + \frac{1}{6}(v-v_c)^3\cdot\left.\frac{\partial^3 P}{\partial v^3}\right|_{T_c,v_c}.$$

Now $\left.\frac{\partial^3 P}{\partial v^3}\right|_{T_c v_c} = -P_c\frac{\partial}{\partial v}\left[\frac{2a}{k_BT_cv^3} - \frac{1}{(v-b)^2}\right] = -P_c\left[\frac{6a}{k_BT_cv_c^4} - \frac{2}{(v_c-b)^3}\right]$

$= -P_c/2b^3$. Thus we obtain near $v=v_c$

$$\boxed{P(v,T_c) = P_c\left[1 - \frac{(v-v_c)^3}{12b^3}\right]}$$

Statistical Mechanics: Problem #2

2. Consider a system of non-interacting spin 1/2 *fermions* at temperature T, in volume V, and chemical potential μ.

   (a) Show that the average occupation numbers of single-particle states of energy $\mu \pm \Delta$ are related by

$$\langle n(\mu + \Delta) \rangle + \langle n(\mu - \Delta) \rangle = 1$$

   where Δ is any constant energy.

Now for the rest of this problem, assume that the single particle states come in pairs of positive and negative energies,

$$\epsilon_{\pm}(\boldsymbol{k}) = \pm \hbar c |\boldsymbol{k}|$$

independent of spin, and that at $T = 0$ all negative energy states are occupied and all positive energy states are empty.

   (b) What is the chemical potential of this system at $T = 0$? Use the result from (a) to show that μ does not change with T.

   (c) Show that the mean total excitation energy of this system at non-zero temperature satisfies

$$E(T) - E(0) = 4V \int_0^{\infty} \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{\epsilon_+(\boldsymbol{k})}{\exp[\beta \epsilon_+(\boldsymbol{k})] + 1} \quad \text{where} \quad \beta \equiv \frac{1}{k_B T} .$$

   (d) Evaluate the heat capacity C_V of this system, using

$$\int_0^{\infty} dx \frac{x^3}{e^x + 1} = \frac{7\pi^4}{120}$$

Solution to S.M. #2

(a) The average occupation number of a state of energy ε is

$$\langle n(\varepsilon)\rangle = \frac{1}{e^{\beta(\varepsilon-\mu)}+1}, \quad \text{with } \beta \equiv (k_B T)^{-1}.$$

Thus $$\langle n(\mu+\Delta)\rangle + \langle n(\mu-\Delta)\rangle = \frac{1}{e^{\beta\Delta}+1} + \frac{1}{e^{-\beta\Delta}+1}$$

$$= \frac{2+e^{\beta\Delta}+e^{-\beta\Delta}}{2+e^{\beta\Delta}+e^{-\beta\Delta}} = 1 \quad \text{as advertized.}$$

(b) At $T=0$, the chemical potential must be at the border between filled and empty states. Thus $\mu(T=0)=0$.

Now assume $\mu=0$ for $T\neq 0$. From above, we would have

$$\overset{\text{spin}}{2}\sum_{k}\left(\frac{1}{e^{\beta ck|k|}+1} + \frac{1}{e^{-\beta ck|k|}+1}\right)$$ is independent of temperature and thus

also equal to $2\sum_{k}\left(\frac{1}{e^{\infty ck|k|}+1} + \frac{1}{e^{-\infty ck|k|}+1}\right)$ which corresponds

to $\mu(T=0)=0$. Thus μ must be zero for all T. Because of the symmetry between the positive and negative energy states the chemical potential is pinned at $\mu=0$.

(c) Subtraction of the energy of occupied states at $T=0$ gives

$$E(T)-E(0) = \sum_{k,\sigma} \Big[\langle n_+(k)\rangle \varepsilon_+(k) + \langle n_-(k)\rangle \varepsilon_-(k) - \varepsilon_-(k) \Big]$$

but $\langle n_-(k)\rangle = 1 - \langle n_+(k)\rangle$ and $\varepsilon_-(k) = -\varepsilon_+(k)$. Thus

$$E(T)-E(0) = \overset{\text{spin}}{2} \sum_{k} \Big[2\langle n_+(k)\rangle \varepsilon_+(k) \Big]$$

In the large V limit $\sum_k \to \frac{V}{(2\pi)^3}\int_0^\infty d^3k$ and thus

$$\boxed{E(T)-E(0) = 4V \int_0^\infty \frac{d^3k}{(2\pi)^3} \frac{\varepsilon_+(k)}{e^{\beta\varepsilon_+(k)}+1}}$$

(d) Using $\varepsilon_+(k) = \hbar c k$ we obtain

$$E(T)-E(0) = 4V \int_0^\infty \frac{4\pi k^2 k}{8\pi^3} \frac{\hbar c k}{\exp(\beta\hbar c k)+1}.$$ Now set $x = \beta\hbar c k$

$$= \frac{2V}{\pi^2} k_B T \left(\frac{k_B T}{\hbar c}\right)^3 \int_0^\infty dx \frac{x^3}{e^x+1} = \frac{7\pi^2}{60} V k_B T \left(\frac{k_B T}{\hbar c}\right)^3$$

The heat capacity is then

$$\boxed{C_V = \frac{dE}{dT} = \frac{7\pi^2}{15} V k_B \left(\frac{k_B T}{\hbar c}\right)^3}$$

Quantum Mechanics: Problem #1

1. Consider a particle of charge q and mass m confined to the x-y plane and subject to a harmonic oscillator potential $V = \frac{1}{2}mw^2(x^2 + y^2)$ and a uniform electric field of magnitude E oriented along the positive x-direction.

   (a) What is the Hamiltonian for the system?

   (b) What are the eigenvalues and associated degeneracies for this Hamiltonian?

Now assume the particle is in the ground state of this system.

   (c) If the electric field is suddenly turned off, show that the probability of finding the particle with energy $(n+1)\hbar w$ is given by a Poisson distribution

$$P(n) = \frac{e^{-\lambda}\lambda^n}{n!}$$

   Find λ.

[You may find the following expressions useful:

$$a_x^+ = \left(\frac{1}{2mw\hbar}\right)^{1/2} (mwx - ip_x)$$

and $\exp[A + B] = \exp(A)\exp(B)\exp\left(-\frac{1}{2}[A, B]\right)$ where $[A, B]$ is a c-number.]

a) $\mathcal{H} = p_x^2/2m + p_y^2/2m + \frac{1}{2}m\omega^2(x^2+y^2) - qEx$

(b) $\mathcal{H} = \mathcal{H}_x + \mathcal{H}_y$, with $\Psi = \Psi_x \Psi_y$

$\mathcal{H}_y = p_y^2/2m + \frac{1}{2}m\omega^2 y^2 \longrightarrow (n_y + 1/2)\hbar\omega$; $\Psi_y \to |n_y\rangle$

$\mathcal{H}_x = p_x^2/2m + \frac{1}{2}m\omega^2\left(x - \frac{qE}{m\omega^2}\right)^2 - \frac{1}{2}\frac{q^2E^2}{m\omega^2} \to (n_x+\frac{1}{2})\hbar\omega - \frac{1}{2}\frac{q^2E^2}{m\omega^2}$

translation operator

$\therefore$ with $\Psi_x \to |\tilde{n}_x\rangle$ where $|\tilde{n}_x\rangle = T(a)|n_x\rangle$
with $a = \frac{qE}{m\omega^2}$

$$\boxed{E_n^{TOT} = (n+1)\hbar\omega - \frac{q^2E^2}{2m\omega^2}}$$

with $n = n_x + n_y$
and therefore degeneracy $\boxed{n+1}$
$\{ |\widetilde{n-n_y}\rangle |n_y\rangle \}$

(c) Particle is in state $|\tilde{n}_x = 0\rangle |n_y = 0\rangle$; $n = 0$.
With $E = 0$, the states of the system are $|n_x\rangle|n_y\rangle$
Thus we need to calculate only $|\langle n_x | \tilde{n}_x = 0\rangle|^2$
and $n_y = 0$, Now:

$|\tilde{0}_x\rangle = e^{-i\left(\frac{qE}{m\omega^2\hbar}\right)p_x}|0_x\rangle$ with $p_x = \left(\frac{\hbar m\omega}{2}\right)^{1/2}(a_x - a_x^+)(i)$

Ignore x-subscripts in what follows.

$|\tilde{0}\rangle = e^{-\left(\frac{q^2E^2}{2m\omega^3\hbar}\right)^{1/2}(a-a^+)}|0\rangle \equiv e^{-\lambda^{1/2}(a-a^+)}|0\rangle$

where we've defined $\boxed{\lambda \equiv q^2E^2/2m\omega^3\hbar}$

Now $e^{-\lambda^{1/2}(a-a^{+})} = e^{-\lambda^{1/2}a}\, e^{\lambda^{1/2}a^{+}}\, e^{\frac{1}{2}\lambda[a,a^{+}]}$, $[a,a^{+}] = 1$ and $\langle n| = \langle 0| \frac{a^{n}}{\sqrt{n!}}$

So $\langle n|\tilde{0}\rangle = \langle 0| \frac{a^{n}}{\sqrt{n!}} \sum_{l=0}^{\infty} \frac{1}{l!}(-\lambda^{1/2}a)^{l} \sum_{m=0}^{\infty} \frac{1}{m!}(\lambda^{1/2}a^{+})^{m}\, e^{\lambda/2}|0\rangle$.

but $m = l+n$. Thus

$$\langle n|\tilde{0}\rangle = \frac{1}{\sqrt{n!}} \lambda^{n/2} \sum_{l=0}^{\infty} \frac{1}{l!}\cdot(-1)^{l}\lambda^{l} \frac{1}{\cancel{(l+n)!}} \langle 0| a^{l+n} \cancel{(a^{+})^{l+n}} |0\rangle \cdot e^{\lambda/2}, \quad \cancel{(l+n)!}$$

$$\langle n|\tilde{0}\rangle = \frac{1}{\sqrt{n!}} \lambda^{n/2}\, e^{-\lambda} \cdot e^{\lambda/2} = \frac{1}{\sqrt{n!}} \lambda^{n/2}\, e^{-\lambda/2}$$

$$\boxed{|\langle n|\tilde{0}\rangle|^{2} = \frac{1}{n!}\cdot\lambda^{n} e^{-\lambda}} \qquad \boxed{\lambda = \frac{q^{2}E^{2}}{2m\omega^{3}\hbar}}$$

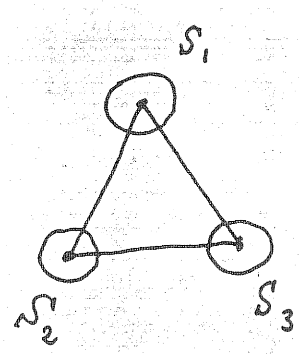
Quantum Mechanics: Problem #2

2. A molecule consisting of three fixed identical atoms in an equilateral triangle captures an extra electron. Assume that in this system the electron is described by a Hamiltonian H. Ignore the spin of the electron and any other electrons already present in the atoms. To obtain the eigenstates of this captured electron we use a simple basis set consisting of one spherically symmetric localized orbital $|S_i\rangle$ on each atom i and assume that they are orthonormal (i.e., $\langle S_i|S_j\rangle = \delta_{ij}$).

   (a) Suppose that $\langle S_1|H|S_2\rangle = \langle S_2|H|S_3\rangle = \langle S_3|H|S_1\rangle \equiv V$ are the *only* non-zero matrix elements in this basis set, find the energies for the captured electron.
   [Hint: Use the fact that one of the eigenvalues is $2V$, which is non-degenerate.]

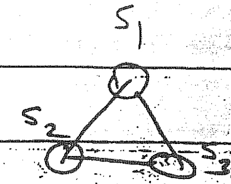
   (b) Since H is invariant under rotations by $\frac{2\pi}{3}$, construct all simultaneous eigenstates of energy and rotation for the captured electron. What are the rotational eigenvalues for each eigenstate?
   [Hint: The rotation operator $R\left(\frac{2\pi}{3}\right)$ satisfies $R^3 = 1$.]

   (c) Suppose that at time $t = 0$ the electron is captured completely by atom #1 in the state $|S_1\rangle$. What is the probability of finding this electron on atom #1 at a later time t? Describe the motion of this electron.



(a) we need to diagonalize a 3×3 matrix

since $\psi = \sum_{i=1}^{3} c_i |s_i\rangle$



$\delta\left[\langle\psi|H|\psi\rangle - E\langle\psi|\psi\rangle\right] = 0 \Rightarrow$

$$\det\begin{pmatrix} -E & V & V \\ V & -E & V \\ V & V & -E \end{pmatrix} = 0 \Rightarrow E^3 - 3EV^2 - 2V^3 = 0$$

since one solution is $E = 2V$, as given in the problem, we need to solve for $E^2 + 2EV + V^2 = 0 \Rightarrow E = -V$ (doubly degenerate)

(b) Since the eigenstate $E = 2V$ is non-degenerate, the state is already an eigenfunction of the rotation operator. Setting $E = 2V$ above, gives:

$$\begin{aligned} -2c_1 + c_2 + c_3 &= 0 \\ c_1 - 2c_2 + c_3 &= 0 \\ c_1 + c_2 - 2c_3 &= 0 \end{aligned} \Rightarrow c_1 = c_2 = c_3 = \frac{1}{\sqrt{3}}$$

$$\therefore \boxed{\psi_{2V} = \frac{1}{\sqrt{3}}\left(|s_1\rangle + |s_2\rangle + |s_3\rangle\right)}$$

clearly $R\left(\frac{2\pi}{3}\right)\psi_{2V} = \frac{1}{\sqrt{3}}\left(|s_2\rangle + |s_3\rangle + |s_1\rangle\right) = \lambda \cdot \psi_{2V}$, with $\lambda = 1$.

What are the other possible eigenvalues of R? Since $R^3 = 1 \Rightarrow \lambda = 1, e^{i2\pi/3}, e^{-i2\pi/3}$

Since the degenerate solutions at $E = -V$ can be chosen to be simultaneous eigenfunctions of R, they must correspond identically to the eigenfunctions of R at $\lambda_+ = e^{i 2\pi/3}$, and $\lambda_- = e^{-2\pi/3}$.

But $R(\frac{2\pi}{3}) |\psi_\alpha\rangle = c_1^{(\alpha)}|s_2\rangle + c_2^{(\alpha)}|s_3\rangle + c_3^{(\alpha)}|s_1\rangle$

$= \lambda_\alpha \left(c_1^{(\alpha)}|s_1\rangle + c_2^{(\alpha)}|s_2\rangle + c_3^{(\alpha)}|s_3\rangle \right)$

$\Rightarrow \quad c_2^{(\alpha)} = c_1^{(\alpha)}/\lambda_\alpha, \quad c_3^{(\alpha)} = c_2^{(\alpha)}/\lambda_\alpha = c_1^{(\alpha)}/\lambda_\alpha^2$

and $c_1^{(\alpha)} = 1/\sqrt{3}$ for normalization. Thus,

$$\boxed{\begin{aligned} \psi_{-V}^{(+)} &= \frac{1}{\sqrt{3}}\left(|s_1\rangle + e^{-i2\pi/3}|s_2\rangle + e^{i2\pi/3}|s_3\rangle\right) \quad \text{for } \lambda_+ = e^{i2\pi/3} \\ \psi_{-V}^{(-)} &= \frac{1}{\sqrt{3}}\left(|s_1\rangle + e^{i2\pi/3}|s_2\rangle + e^{-i2\pi/3}|s_3\rangle\right) \quad \text{for } \lambda_- = e^{-i2\pi/3} \end{aligned}}$$

c) $|s_1\rangle = |\psi_{2V}\rangle\langle\psi_{2V}|s_1\rangle + |\psi_{-V}^+\rangle\langle\psi_{-V}^+|s_1\rangle + |\psi_{-V}^-\rangle\langle\psi_{-V}^-|s_1\rangle$

$|s_1\rangle = \frac{1}{\sqrt{3}}\left(|\psi_{2V}\rangle + |\psi_{-V}^+\rangle + |\psi_{-V}^-\rangle\right)$

$|s_1(\tau)\rangle = \frac{1}{\sqrt{3}}\left(e^{-i2V\tau/\hbar}|\psi_{2V}\rangle + e^{iV\tau/\hbar}\left(|\psi_{-V}^+\rangle + |\psi_{-V}^-\rangle\right)\right)$

$\langle s_1|s_1(\tau)\rangle = \frac{1}{3}\left(e^{-i2V\tau/\hbar} + 2e^{iV\tau/\hbar}\right)$

$$\boxed{|\langle s_1|s_1(\tau)\rangle|^2 = \frac{1}{9}\left(5 + 4\cos\left(3V\tau/\hbar\right)\right)}$$

electron cycles around the ring with frequency $3V/\hbar$.