# DOCTORAL GENERAL EXAMINATION

## PART II

### February 9, 2001

### FIVE HOURS

1. This examination is divided into four sections, Mechanics, Electricity & Magnetism, Statistical Mechanics, and Quantum Mechanics, with two problems in each. Read both problems in each section carefully before making your choice. Submit ONLY one problem per section. IF YOU SUBMIT MORE THAN ONE PROBLEM FROM A SECTION, BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.

2. Use a separate paper (i.e. folded sheet) for each problem. Write your name and the problem number (I.2 for example) on each folded sheet.

3. Calculators may be used.

4. No books or reference materials may be used.

# Mechanics Problem 1

Consider a system made of two rigid bodies. The first body of mass $m_1$ is pivoted about point $A$ as shown in the diagram. Its center of mass is at $D$ and its moment of inertia about $D$ is $I$. The second body is a massless rigid rod with one end pivoted about point $B$ on the first body. At the other end of the massless rod is mass $m_2$. Note that point $D$ is on the line joining $A$ and $B$. The system is moving in a plane. Let $AD = a$, $AB = b$ and $BC = c$. The gravitational acceleration is $g$.
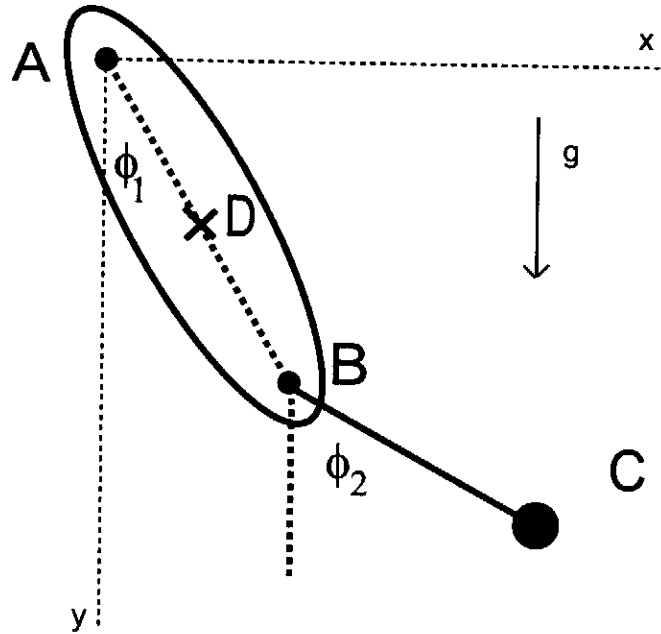


a) Write Lagrangian for the system as a function of coordinates $\phi_1$ and $\phi_2$.

b) Write down the equations of motion for the system. Do not solve.

c) Find a condition for the masses of the two pendula $m_1$, $m_2$ the moment of inertia $I$ as well as the geometrical parameters $a$, $b$, $c$ such that the system can move as one rigid body ($\phi_1 = \phi_2$).

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# Mechanics Problem 2

Three identical masses $m$ are contrained to move on a circle. The masses are connected with identical springs each with spring constant $k$ as shown in the figure. There is no friction and no gravity.
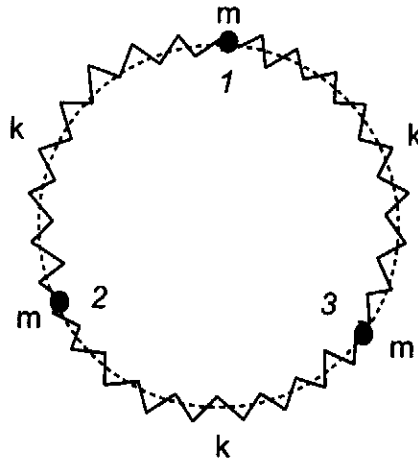


a) Find all the normal frequencies of oscillations of the system.

b) Find the normal coordinates of the system in terms of the displacements of individual masses $x_i$ where $i = 1, 2, 3$.

c) At $t = 0$ mass at position 1 was found to be displaced by a distance $x_1 = \delta$ from its equilibrium position. The velocities of all the three masses at $t = 0$ were 0. Find the motion of the three masses as a function of time.

# Electromagnetism Problem 1

A plane electromagnetic wave of angular frequency $\omega$ is propagating through an optically active medium. The polarization vector of the medium is given by $\vec{P} = \gamma \vec{\nabla} \times \vec{E}$ where $\gamma$ is a real constant and $\gamma c \mu_0 \omega \ll 1$. The medium is non-conductive and non-magnetic. Assume that the wave is moving along the $z$-axis.

Useful definitions and equations:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \qquad \vec{B} = \mu_0(\vec{H} + \vec{M})$$

for $\rho = 0$ and $\vec{j} = 0$ the Maxwell equations are:

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \cdot \vec{B} = 0$$

a) The wave travelling in this medium will see two different refraction coefficients. Find these two coefficients

b) Find the electric field configurations that correspond to the two different refraction coefficients.

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# Electromagnetism Problem 2

Consider a perfectly conducting sphere of radius $R$. The sphere is isolated and it is charged with an initial charge $Q_0$ ($Q_0$ can be positive or negative). An external uniform electric field is applied to the sphere. At an infinite distance from the sphere the field is uniform and parallel to the $z$-axis $\vec{E}(\vec{r}) = E_0 \hat{z}$.

The general solution for an axially symmetric potential in spherical coordinates is

$$V(r,\theta) = \sum_{n=0}^{\infty} (A_n r^n + B_n r^{-(n+1)}) P_n(\cos\theta) \quad \text{and} \quad P_n(\beta = \cos\theta) = \frac{1}{2^m m!} \frac{d^m}{d\beta_m} (\beta^2 - 1)^m$$

also, in spherical coordinates

$$\vec{\nabla} = \hat{r} \frac{\partial}{\partial r} + \hat{\phi} \frac{1}{r\sin\theta} \frac{\partial}{\partial \phi} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta}$$

a) Find the electric field $\vec{E}(r,\theta)$ outside of the sphere ($r > R$) as a function of $E_0$, $R$, $r$, $\theta$, $Q_0$ and physical constants.

b) Use results of a) to find the region on the sphere where the radial component of the electric field at the surface is pointing towards the center of the sphere (inwards). Both the existence and the size of the region will depend on the value of $Q_0$; explain.

Consider now a cloud of small positively charged dust particles surrounding the sphere with an approximately constant charge density $\rho_0$ near to the sphere's surface. The mobility of dust particles is $\mu$ ($\vec{v} = \mu \vec{E}$). For a sufficiently small initial charge $Q_0 < Q_{max}$ the dust will be settling on the sphere increasing its original charge and resulting in time dependent charge on the sphere $Q(t)$. Assume that the charge of the dust cloud is small such that it does not affect the electric field other than by charging up the sphere. Assume the dust cloud density does not change in time, and that the motion of charges is so slow that magnetic effects can be ignored.

c) Find a maximum initial charge $Q_0 = Q_{max}$ for the dust to be able to reach the surface.

d) Assuming that $Q_0 < Q_{max}$ calculate the radial current density of the dust falling on the surface of the sphere as a function of polar angle $\theta$.

e) Find the rate of change of the charge of the sphere for (1) $Q(t) < -Q_{max}$ and for (2) $-Q_{max} < Q(t) < Q_{max}$ as a function of given parameters and $Q(t)$. To simplify the calculations normalize the charge using value of $Q_{max}$ obtained previously such that the normalized rate is

$$\text{rate} = \frac{1}{Q_{max}} \frac{dQ}{dt}.$$

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# Quantum Mechanics Problem 1

Consider a spin $\frac{1}{2}$ particle where $\hat{S}_j = \frac{\hbar}{2}\sigma_j$ and

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

a) The operator for spin along the $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ direction is:

$$\hat{S}_{\vec{n}} = n_x \hat{S}_x + n_y \hat{S}_y + n_z \hat{S}_z.$$

Find the eigenvalues of $\hat{S}_{\vec{n}}$ and the associated eigenvectors.

b) Is it possible for a spin $\frac{1}{2}$ particle to be in a state $|\psi>$ such that:

$$< \psi|\hat{S}_x|\psi > = < \psi|\hat{S}_y|\psi > = < \psi|\hat{S}_z|\psi > = 0?$$

If it is possible, find $|\psi>$. If it is not, show why.

c) A beam of spin $\frac{1}{2}$ particles enters a Stern-Gerlach filter which allows only particles whose spin is $+\frac{1}{2}\hbar$ along the $z$ direction to pass. The particles then pass through a region where there is magnetic field $\vec{B} = (0, B, 0)$ and spend time $T$ in this region. Then the particles enter a Stern-Gerlach filter which allows only particles whose spin is $+\frac{1}{2}\hbar$ along the $x$ direction to pass. What fraction of the electrons which exit the first filter exit the second? Assume that the particles interact with the magnetic field via:

$$\hat{H} = -\sum_j B_j \hat{S}_j$$

# Quantum Mechanics Problem 2

A particle of mass $m = 1$ is in the ground state of a harmonic oscillator with Hamiltonian

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{x}^2).$$

The state is translated a distance $d$ while $\hat{H}$ is left alone.

a) What is the probability that a measurement of the energy will yield the value $\hbar\omega(j + \frac{1}{2})$? You may find the following formulas useful:

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{x} + i\hat{p}) \qquad |n> = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0>$$

and also if $\hat{A}$ and $\hat{B}$ are two operators such that $[\hat{A}, \hat{B}]$ is proportional to a unit operator, then

$$e^{(\hat{A}+\hat{B})} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]}.$$

At $t = 0$ the quantum system is prepared in the translated state described above. It then evolves according to the Schrödinger equation with the Hamiltonian $\hat{H}$.

b) Supose that at the time $t$ the momentum is measured. At what values of $t$ will the probability distribution for getting any value of the momentum be identical to the $t = 0$ probability distribution?

c) repeat b) with momentum measurement replaced by energy measurement.

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# Statistical Mechanics Problem 1

Consider an ideal gas where $pV = NkT$ and $U = \frac{f}{2}NkT$ where $f$ is the number of degrees of freedom of the gas.

a) The Carnot Engine is an ideal gas in a cylinder with a piston. It goes through a four step reversible process:

   1) The gas expands isothermally at temperature $T_H$ from volume $V_a$ to volume $V_b$. Heat $|Q_H|$ is delivered to the system.

   2) The gas is thermally isolated and expands adiabatically from $V_b$ to $V_c$ cooling to temperature $T_L$.

   3) The gas is compressed isothermally at $T_L$ from volume $V_c$ to $V_d$. Heat $|Q_L|$ is given off.

   4) The gas is compressed adiabatically back to temperature $T_H$ and volume $V_a$.

   Let the efficiency be given by

$$e = \frac{|Q_H| - |Q_L|}{|Q_H|}.$$

   Compute $e$ in terms of $T_L$ and $T_H$.

b) Find a relation between $C_p$ and $C_V$ involving the gas dependent constant $f$ for an ideal gas.

   Here $C_p$ is the heat capacity at constant pressure:

$$C_p = \left( \frac{\Delta Q}{\Delta T} \right)_p$$

   and $C_V$ is the heat capacity at constant volume:

$$C_V = \left( \frac{\Delta Q}{\Delta T} \right)_V.$$

# Statistical Mechanics Problem 2

Consider a zero temperature gas of non-relativistic electrons. The electrons are spin $\frac{1}{2}$ fermions with up/down degeneracy.

a) Ignore the interaction between the electrons and calculate the Fermi wave number $k_f$ and the ground state energy density $E/V$ in terms of the number density $n$.

Assume now that the Coulomb repulsion along with the exclusion principle results in an additional term in the Hamiltonian:

$$U = \alpha \frac{N_+ N_-}{V}$$

where $\alpha$ is a constant and $N_+$ is the number of electrons with spin up and $N_-$ is the number with spin down.

b) Express the modified Fermi wave numbers $k_{f+}$ and $k_{f-}$ in terms of $n_+ = N_+/V$ and $n_- = N_-/V$.

c) Assuming small deviations $n_+ = \frac{n}{2} + \delta$ and $n_- = \frac{n}{2} - \delta$ from the symmetric state, calculate the change in kinetic energy to second order in $\delta$.

d) Express $U/V$ in terms of $\delta$.

e) Find the critical value $\alpha_c$ such that for $\alpha > \alpha_c$ the electron gas can lower its energy by spontaneously developing magnetization.

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