2000 Spring General Exam Part II Solutions for CM and E&M

Typeset by Mingyu (Charles) Li*

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1. Classical Mechanics

Problem 1

(a)

For a three-atom molecule there are 9 degrees of freedoms in total (3 for each atom). We subtract the 3 translational DOF and 2 rotational DOF, and the number of vibrational DOF left is 4. 2 of them are longitudinal and 2 of them are transverse, but the transverse ones have the same frequency by cylindrical symmetry of the potential, and thus we can constrain ourselves to the discussion of $x-y$ plane, with 3 effective modes.

(b)

From the condition that total $P_x = P_y = 0$:

$$m_{\rm A}\dot{x}_1 + m_{\rm B}\dot{x}_2 + m_{\rm A}\dot{x}_3 = 0, \tag{1}$$

$$m_{\rm A}\dot{y}_1 + m_{\rm B}\dot{y}_2 + m_{\rm A}\dot{y}_3 = 0. \tag{2}$$

Also from $J_z = 0$, and putting $\overline{A_1B} = \overline{BA_2} = l$:

$$J_z = m_{\rm A}l[(\dot{y}_2 - \dot{y}_1) + (\dot{y}_3 - \dot{y}_2)] = m_{\rm A}l(\dot{y}_3 - \dot{y}_1) = 0. \tag{3}$$

Thus,

$$m_{\rm A}x_1 + m_{\rm B}x_2 + m_{\rm A}x_3 = 0, \tag{4}$$

$$m_{\rm A}y_1 + m_{\rm B}y_2 + m_{\rm A}y_3 = 0, \tag{5}$$

$$y_3 - y_1 = 0, \tag{6}$$

*mingyuli@mit.edu

by the fact that the integrations yield constants and at equilibrium x_i and y_i are 0. We then get:

$$y_3 = y_1 \equiv y, \tag{7}$$
$$x_2 = -\frac{m_\mathrm{A}(x_1 + x_3)}{m_\mathrm{B}}, \tag{8}$$
$$y_2 = -\frac{2m_\mathrm{A} y}{m_\mathrm{B}}. \tag{9}$$

(c)

We begin by expanding the potential around the equilibrium values:

$$\begin{aligned}
U &= U_\mathrm{eq} + \frac{\kappa}{2}(\delta\overline{A_1B})^2 + \frac{\kappa}{2}(\delta\overline{BA_2})^2 + \frac{\alpha l^2}{2}(\delta\angle A_1BA_2)^2 && (10)\\
&= U_\mathrm{eq} + \frac{\kappa}{2}\Big[(x_1 - x_2)^2 + (x_2 - x_3)^2\Big] + \frac{\alpha l^2}{2}\frac{(y_3 - y_2 + y_1 - y_2)^2}{l^2} && (11)\\
&= U_\mathrm{eq} + \frac{\kappa}{2m_\mathrm{B}^2}\Big\{[m_\mathrm{B}x_1 + m_\mathrm{A}(x_1 + x_3)]^2 + [m_\mathrm{B}x_3 + m_\mathrm{A}(x_1 + x_3)]^2\Big\}\\
&\quad + \frac{\alpha}{2m_\mathrm{B}^2}\Big[4y^2(m_\mathrm{B} + 2m_\mathrm{A})^2\Big]. && (12)
\end{aligned}$$

The change in potential can be written as:

$$\delta U = \vec{\boldsymbol{X}} M_U \vec{\boldsymbol{X}}^T, \tag{13}$$

with:

$$\vec{\boldsymbol{X}} = \begin{bmatrix} x_1 \\ x_3 \\ y \end{bmatrix}, \tag{14}$$

and

$$M_U = \frac{1}{2m_\mathrm{B}^2}\begin{bmatrix} \kappa[m_\mathrm{A}^2 + (m_\mathrm{A} + m_\mathrm{B})^2] & 2\kappa m_\mathrm{A}(m_\mathrm{A} + m_\mathrm{B}) & 0 \\ 2\kappa m_\mathrm{A}(m_\mathrm{A} + m_\mathrm{B}) & \kappa[m_\mathrm{A}^2 + (m_\mathrm{A} + m_\mathrm{B})^2] & 0 \\ 0 & 0 & 4\alpha(m_\mathrm{B} + 2m_\mathrm{A})^2 \end{bmatrix}. \tag{15}$$

The kinetic energy can be calculated in a similar way:

$$T = \frac{1}{2}m_\mathrm{A}(\dot{x}_1^2 + \dot{y}^2) + \frac{1}{2}m_\mathrm{A}(\dot{x}_3^2 + \dot{y}^2) + \frac{1}{2}m_\mathrm{B}\left[\left(\frac{m_\mathrm{A}}{m_\mathrm{B}}\right)^2(\dot{x}_1 + \dot{x}_3)^2 + \left(\frac{4m_\mathrm{A}^2}{m_\mathrm{B}^2}\right)\dot{y}^2\right] \tag{16}$$
$$\Rightarrow T = \dot{\vec{\boldsymbol{X}}} M_T \dot{\vec{\boldsymbol{X}}}^T, \tag{17}$$

with:

$$M_T = \frac{1}{2m_\mathrm{B}^2}\begin{bmatrix} m_\mathrm{A}m_\mathrm{B}(m_\mathrm{A} + m_\mathrm{B}) & m_\mathrm{A}^2 m_\mathrm{B} & 0 \\ m_\mathrm{A}^2 m_\mathrm{B} & m_\mathrm{A}m_\mathrm{B}(m_\mathrm{A} + m_\mathrm{B}) & 0 \\ 0 & 0 & 2m_\mathrm{A}m_\mathrm{B}(m_\mathrm{B} + 2m_\mathrm{A}) \end{bmatrix}. \tag{18}$$

(d)

The Lagrangian is $L = T - U$, and in vector notation, the equation of motion is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\vec{\boldsymbol{X}}}}\right) = 2M_T \ddot{\vec{\boldsymbol{X}}} = \frac{\partial L}{\partial \vec{\boldsymbol{X}}} = -2M_U \vec{\boldsymbol{X}}. \tag{19}$$

The eigenvalues ω satisfy:

$$(M_U - \omega^2 M_T)\vec{\boldsymbol{A}} = \vec{\boldsymbol{0}}. \tag{20}$$

We would like to solve for the determinant:

$$\begin{vmatrix} \chi & \eta & 0 \\ \eta & \chi & 0 \\ 0 & 0 & 4\alpha(m_\mathrm{B} + 2m_\mathrm{A})^2 - 2\omega^2 m_\mathrm{A} m_\mathrm{B}(m_\mathrm{B} + 2m_\mathrm{A}) \end{vmatrix} = 0, \tag{21}$$

with:

$$\chi = \kappa[m_\mathrm{A}^2 + (m_\mathrm{A} + m_\mathrm{B})^2] - \omega^2 m_\mathrm{A} m_\mathrm{B}(m_\mathrm{A} + m_\mathrm{B}), \tag{22}$$
$$\eta = 2\kappa m_\mathrm{A}(m_\mathrm{A} + m_\mathrm{B}) - \omega^2 m_\mathrm{A}^2 m_\mathrm{B}. \tag{23}$$

The three solutions are given by:

$$4\alpha(m_\mathrm{B} + 2m_\mathrm{A})^2 - 2\omega^2 m_\mathrm{A} m_\mathrm{B}(m_\mathrm{B} + 2m_\mathrm{A}) = 0 \Rightarrow \omega_1 = \sqrt{\frac{2\alpha(2m_\mathrm{A} + m_\mathrm{B})}{m_\mathrm{A} m_\mathrm{B}}}, \tag{24}$$
$$\chi = \eta \Rightarrow \omega_2 = \sqrt{\frac{\kappa}{m_\mathrm{A}}}, \tag{25}$$
$$\chi = -\eta \Rightarrow \omega_3 = \sqrt{\frac{\kappa(2m_\mathrm{A} + m_\mathrm{B})}{m_\mathrm{A} m_\mathrm{B}}}. \tag{26}$$

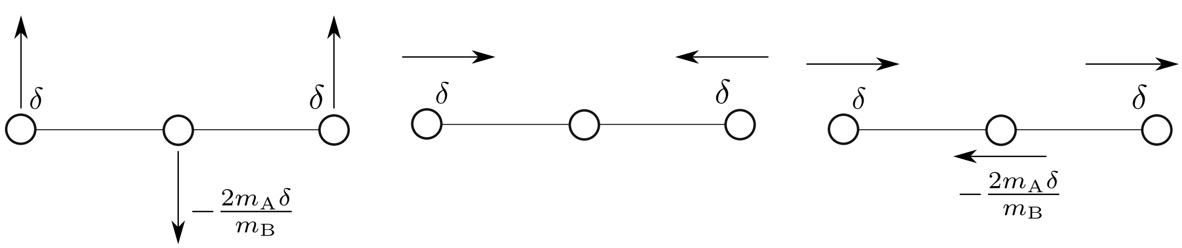

Figure 1: Vibrational modes of the molecule.

The corresponding eigenvectors can be obtained by plugging in matrix $M = M_U - \omega^2 M_T$

and find vectors $\vec{\boldsymbol{A}}_i$ that give $\vec{\mathbf{0}}$:

$$M\vec{\boldsymbol{A}}_1 = \begin{bmatrix} C_1 & C_2 & 0 \\ C_2 & C_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{\boldsymbol{A}}_1 = \vec{\mathbf{0}} \Rightarrow \vec{\boldsymbol{A}}_1 = \begin{bmatrix} 0 \\ 0 \\ \delta \end{bmatrix}, \tag{27}$$

$$M\vec{\boldsymbol{A}}_2 = \begin{bmatrix} D & D & 0 \\ D & D & 0 \\ 0 & 0 & E \end{bmatrix} \vec{\boldsymbol{A}}_2 = \vec{\mathbf{0}} \Rightarrow \vec{\boldsymbol{A}}_2 = \begin{bmatrix} \delta \\ -\delta \\ 0 \end{bmatrix}, \tag{28}$$

$$M\vec{\boldsymbol{A}}_3 = \begin{bmatrix} F & -F & 0 \\ -F & F & 0 \\ 0 & 0 & G \end{bmatrix} \vec{\boldsymbol{A}}_3 = \vec{\mathbf{0}} \Rightarrow \vec{\boldsymbol{A}}_3 = \begin{bmatrix} \delta \\ \delta \\ 0 \end{bmatrix}. \tag{29}$$

Figure 1 shows the corresponding vibrational modes of the molecule.

Problem 2

(a)

The differential cross section is defined as:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\text{\# of particles going through solid angle d}\Omega\text{ per unit time}}{\text{Particle flux (\# of incoming particles per unit time per unit area)}}. \tag{30}$$

(b)

Since the force is a central force, there is no azimuthal angle dependence of the differential cross section, and by conservation of particle numbers:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)\mathrm{d}\Omega = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)2\pi\sin\chi\mathrm{d}\chi = \frac{I2\pi\rho\mathrm{d}\rho}{I} \tag{31}$$

$$\Rightarrow \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\rho}{\sin\chi}\left|\frac{\mathrm{d}\rho}{\mathrm{d}\chi}\right|. \tag{32}$$

(c)

Again by the central force, the angular momentum is conserved. Along with total energy conservation, we have the following two conditions:

$$\dot{\phi} = \frac{l}{mr^2}, \tag{33}$$

$$\frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + U = E \Rightarrow \dot{r} = \sqrt{\frac{2(E-U) - l^2/(mr^2)}{m}}. \tag{34}$$

Divide the conditions and we have:

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{l}{r^2\sqrt{2m(E-U) - l^2/r^2}}. \tag{35}$$

The angular momemtum is a function of the impact parameter ρ:

$$l = mv\rho = \sqrt{2mE}\rho, \tag{36}$$

and thus we get:

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{\rho}{r^2\sqrt{1 - \frac{U}{E} - \frac{\rho^2}{r^2}}}. \tag{37}$$

(d)

We denote the angle of the closest approach to be θ, and there is $\chi = \pi - 2\theta$. We would like to find this angle $\theta(\rho)$ and then find ρ as a function of χ.

Integrating both sides of $\mathrm{d}\phi/\mathrm{d}r$:

$$\theta = \int_{r_{\min}}^{\infty} \frac{\rho}{r\sqrt{r^2 - \frac{Ur^2}{E} - \rho^2}}. \tag{38}$$

The potential from $F = kr^{-3}$ is $U = k/(2r^2)$:

$$\theta = \int_{r_{\min}}^{\infty} \frac{\rho}{r\sqrt{r^2 - \left(\frac{k}{2E} + \rho^2\right)}} \equiv \int_{r_{\min}}^{\infty} \frac{\rho}{r\sqrt{r^2 - c^2}} \tag{39}$$

$$= \frac{\rho}{c}\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{c}{r_{\min}}\right)\right]. \tag{40}$$

To find $r_{\min}$, note that using the condition of energy conservation we can set $\dot{r} = 0$:

$$\frac{2mE\rho^2}{2mr_{\min}^2} + \frac{k}{2r_{\min}^2} = E \Rightarrow r_{\min} = \sqrt{\frac{k}{2E} + \rho^2} = c. \tag{41}$$

Therefore:

$$\theta = \frac{\pi\rho}{2c} \tag{42}$$

$$\Rightarrow \frac{\rho}{c} = 1 - \frac{\chi}{\pi} \tag{43}$$

$$\Rightarrow \rho^2 = \frac{k}{2E}\frac{\left(1 - \frac{\chi}{\pi}\right)^2}{1 - \left(1 - \frac{\chi}{\pi}\right)^2} \tag{44}$$

$$\Rightarrow \rho\frac{\mathrm{d}\rho}{\mathrm{d}\chi} = \frac{-k}{2\pi E}\frac{\left(1 - \frac{\chi}{\pi}\right)}{\left[1 - \left(1 - \frac{\chi}{\pi}\right)^2\right]^2}. \tag{45}$$

Finally:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\pi k\left(1 - \frac{\chi}{\pi}\right)}{2E\chi^2(\sin\chi)\left(2 - \frac{\chi}{\pi}\right)^2}. \tag{46}$$

2. Classical Electromagnetism

Problem 1

(a)

The TEM mode has the $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fields both perpendicular to the wave vector $\vec{\boldsymbol{k}}$, and as shown in figure 2, if we denote the surface charge density on the inner cylinder to be σ_a,

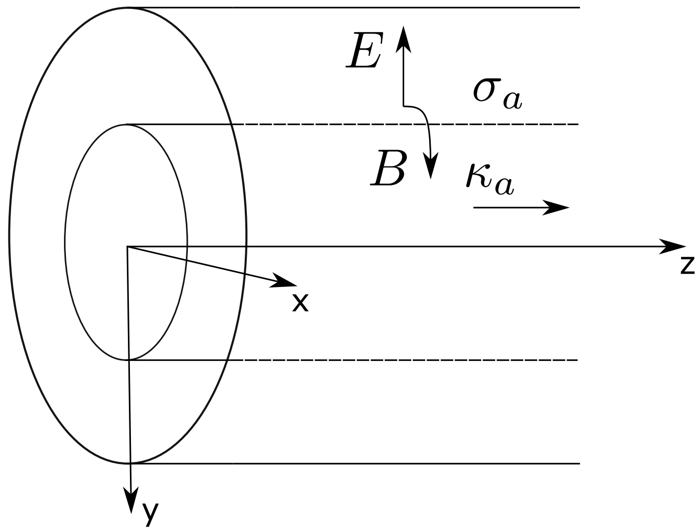

Figure 2: Drawing of the coax cable.

and the current density to be κ_a, the fields are:

$$(2\pi r)(l)(E) = (\sigma_a)(2\pi a)(l)/\epsilon_0 \Rightarrow \vec{\boldsymbol{E}} = \frac{\sigma_a a}{\epsilon_0 r}\hat{\boldsymbol{r}}, \tag{47}$$

$$(2\pi r)B = \mu_0(\kappa_a)(2\pi a) \Rightarrow \vec{\boldsymbol{B}} = \frac{\mu_0 \kappa_a a}{r}\hat{\boldsymbol{\theta}}. \tag{48}$$

For a propagating wave, we use the complex notation:

$$\vec{\boldsymbol{E}} = \frac{\sigma_a a}{\epsilon_0 r} e^{ikz - i\omega t}\hat{\boldsymbol{r}}, \tag{49}$$

$$\vec{\boldsymbol{B}} = \frac{\mu_0 \kappa_a a}{r} e^{ikz - i\omega t}\hat{\boldsymbol{\theta}}. \tag{50}$$

In Gaussian units:

$$\vec{\boldsymbol{E}} = \frac{4\pi \sigma_a a}{r} e^{ikz - i\omega t}\hat{\boldsymbol{r}}, \tag{51}$$

$$\vec{\boldsymbol{B}} = \frac{4\pi \kappa_a a}{cr} e^{ikz - i\omega t}\hat{\boldsymbol{\theta}}. \tag{52}$$

(b)

The voltage V_{ab} can be obtained by integrating the $\vec{\boldsymbol{E}}$ field:

$$|V_{ab}| = \int_a^b \frac{\sigma_a a}{\epsilon_0 r}\mathrm{d}r = \frac{\sigma_a a}{\epsilon_0}\ln\left(\frac{b}{a}\right). \tag{53}$$

On the other hand, we have:

$$|E| = c|B| \Rightarrow \sigma_a/\kappa_a = 1/c, \tag{54}$$

and:

$$|I_{ab}| = 2\pi a\kappa_a. \tag{55}$$

Therefore, we have the effective resistance:

$$R_{\text{eff}} = V_{ab}/I_{ab} = \frac{1}{2\pi c\epsilon_0}\ln\left(\frac{b}{a}\right). \tag{56}$$

In Gaussian units:

$$R_{\text{eff}} = \frac{2}{c}\ln\left(\frac{b}{a}\right). \tag{57}$$

(c)

From Kirchoff's law:

$$I_L + I_C + I_R = 0 \Rightarrow \frac{V}{L} + C\ddot{V} + \frac{\dot{V}}{R} = 0. \tag{58}$$

We use the ansatz $V = V_0 e^{-i\omega t}$ (the sign of the exponential is important in part (d) for consistency with the voltage expression):

$$\omega^2 + \frac{i\omega}{RC} - \frac{1}{LC} = 0 \tag{59}$$

$$\Rightarrow \omega = \frac{-i}{2RC} \pm \frac{1}{2}\sqrt{\frac{4}{LC} - \frac{1}{R^2C^2}} \equiv -i\gamma \pm \omega_0. \tag{60}$$

The Q factor is:

$$Q = \omega_0/(2\gamma) \approx \frac{1}{\sqrt{LC}}RC = R\sqrt{\frac{C}{L}}. \tag{61}$$

Thus, for Q to be larger than 100, we find:

$$\frac{L}{C} < \frac{10^{-4}}{4\pi^2c^2\epsilon_0^2}\ln^2\left(\frac{b}{a}\right). \tag{62}$$

In Gaussian units:

$$\frac{L}{C} < \frac{4\times 10^{-4}}{c^2}\ln^2\left(\frac{b}{a}\right). \tag{63}$$

(d)

We have both an input wave and a reflected wave in the cable, so the total field magnitudes are:

$$E = \frac{a}{\epsilon_0 r}(\sigma_a^+ e^{ikz-i\omega t} + \sigma_a^- e^{-ikz-i\omega t}), \tag{64}$$

$$B = \frac{\mu_0 a}{r}(\kappa_a^+ e^{ikz-i\omega t} + \kappa_a^- e^{-ikz-i\omega t}). \tag{65}$$

Note that the current densities are in opposite direction. We then apply the boundary condition that the current density must be 0 at the open end, and use the relation $\sigma_a/\kappa_a = 1/c$ to obtain the plus and minus charge and current densities:

$$I \sim (\kappa^+ - \kappa^-) \Rightarrow \kappa_a^- = \kappa_a^+ e^{ik2D} \Rightarrow \sigma_a^- = \sigma_a^+ e^{ik2D}. \tag{66}$$

The effective resistance can be computed as:

$$R_{\text{eff}} = \frac{1}{2\pi c\epsilon_0} \ln\left(\frac{b}{a}\right) \frac{1 + e^{ik2D}}{1 - e^{ik2D}} \equiv iR_0 \cot(kD). \tag{67}$$

To calculate the resonance frequency, we substitute R into equation 59 and let $\omega_0^2 = 1/LC$:

$$\omega^2 - \omega_0^2 = -\frac{\omega \tan(kD)}{R_0 C} \tag{68}$$

$$\Rightarrow \tan(kD) = \frac{C}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right)\left(\frac{k_0^2}{k} - k\right). \tag{69}$$

Plotting both sides on the graph (using unitary constants, such as $k_0 = 1, D = 1$), we note that when $D \to \infty$, the minimum $k \to 0$. See figure 3.

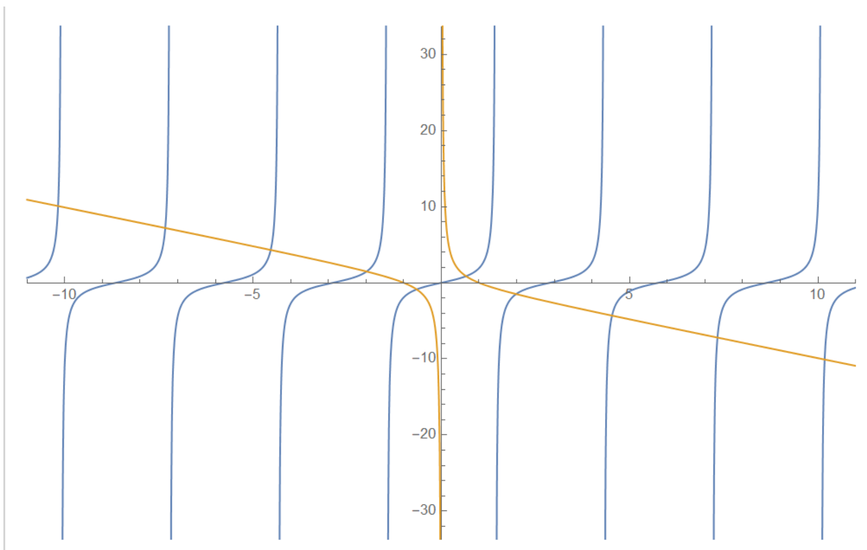

Figure 3: Graph of the characteristic equation for the resonance frequency.

(e)

As $\lambda \gg d$, we use the approximation that the voltages around the junction are equal (because of perfect conductor). We also impose the current continuity condition around the junction. Thus the voltage and current before the junction are related to the voltage and current after the junction by the following:

$$V_+ + V_- \sim \ln\left(\frac{b_1}{a}\right)(\sigma_+ + \sigma_-) = \ln\left(\frac{b_2}{a}\right)\sigma_t \sim V_t, \tag{70}$$

$$I_+ + I_- \sim (\kappa_+ - \kappa_-) = \kappa_t \sim I_t, \tag{71}$$

$$\sigma_a/\kappa_a = 1/c. \tag{72}$$

The reflection coefficient is then:

$$R = \left|\frac{E_{\text{refl}}}{E_{\text{inc}}}\right|^2 = \left|\frac{\kappa_-}{\kappa_+}\right|^2 = \left|\frac{\ln(b_2/a) - \ln(b_1/a)}{\ln(b_2/a) + \ln(b_1/a)}\right|^2. \tag{73}$$

Problem 2

(a)

Since the B-field does not do work, the total energy of the proton is conserved. For a charged-particle in E&M, the total energy is:

$$U = \frac{(\vec{\boldsymbol{p}} - q\vec{\boldsymbol{A}})^2}{2m}, \tag{74}$$

or in Gaussian units:

$$U = \frac{(\vec{\boldsymbol{p}} - q\vec{\boldsymbol{A}}/c)^2}{2m}. \tag{75}$$

Also, the momentum parallel to the B-field (i.e. p_y) is conserved since its cross product with $\vec{\boldsymbol{B}}$ is $\vec{\boldsymbol{0}}$.

(b)

In the $x^2 \gg \lambda^2$ region, the B-field is roughly constant, and thus we expect a small deviation from the regular cyclotron motion. If we write the position of the proton as:

$$\vec{\boldsymbol{\rho}} = \vec{\boldsymbol{R}} + \vec{\boldsymbol{r}}_c, \tag{76}$$

where $\vec{\boldsymbol{r}}_c$ is the fast cyclotron motion, and expand the B-field to first order around its value at the center of the cyclotron motion, then the equation of motion is:

$$m\ddot{\vec{\boldsymbol{\rho}}} = m\ddot{\vec{\boldsymbol{R}}} + m\ddot{\vec{\boldsymbol{r}}}_c \approx m\ddot{\vec{\boldsymbol{r}}}_c = q(\vec{\boldsymbol{v}}_\perp \times \vec{\boldsymbol{B}}) \tag{77}$$

$$\Rightarrow \cancel{m\ddot{\vec{\boldsymbol{r}}}_c} = q[\cancel{\vec{\boldsymbol{v}}_c \times \vec{\boldsymbol{B}}_0} + \vec{\boldsymbol{v}}_d \times \vec{\boldsymbol{B}}_0 + \vec{\boldsymbol{v}}_c \times (\vec{\boldsymbol{r}}_c \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{B}}] \tag{78}$$

$$\Rightarrow q[\vec{\boldsymbol{v}}_d \times \vec{\boldsymbol{B}}_0 + \vec{\boldsymbol{v}}_c \times (\vec{\boldsymbol{r}}_c \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{B}}] = 0. \tag{79}$$

Here $\vec{\boldsymbol{v}}_d$ means the small drift velocity in addition to the cyclotron motion. We average the last line through one cyclotron cycle:

$$\langle \vec{\boldsymbol{v}}_d \rangle \times \vec{\boldsymbol{B}}_0 + \left\langle \vec{\boldsymbol{v}}_c \times (\vec{\boldsymbol{r}}_c \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{B}} \right\rangle = 0. \tag{80}$$

Use:

$$\vec{\boldsymbol{r}}_c = (r\cos(\theta), 0, r\sin(\theta)) \tag{81}$$

$$\vec{\boldsymbol{v}}_c = (-r\sin(\theta)\omega_c, 0, r\cos(\theta)\omega_c) \tag{82}$$

$$\Rightarrow \left\langle \vec{\boldsymbol{v}}_c \times (\vec{\boldsymbol{r}}_c \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{B}} \right\rangle = \langle \vec{\boldsymbol{v}}_c \times (r\cos(\theta)\partial_x B_y \hat{\boldsymbol{y}}) \rangle \tag{83}$$

$$= \cancel{\left\langle -\omega_c r^2 \sin\cos \partial_x B_y \right\rangle} \hat{\boldsymbol{z}} - \left\langle \omega_c r^2 \cos^2 \partial_x B_y \right\rangle \hat{\boldsymbol{x}} = -\frac{1}{2}\omega_c r^2 \partial_x B_y \hat{\boldsymbol{x}}. \tag{84}$$

This is equivalent to an effective E-field of:

$$\vec{\boldsymbol{E}}_{\text{eff}} = -\frac{1}{2}\omega_c r^2 \partial_x B_y \hat{\boldsymbol{x}} = -\frac{v_c^2}{2\omega_c}\partial_x B_y \hat{\boldsymbol{x}}. \tag{85}$$

Thus the drift velocity can be computed as:

$$\vec{\boldsymbol{v}}_d = \frac{\vec{\boldsymbol{E}}_{\text{eff}} \times \vec{\boldsymbol{B}}}{B^2} \tag{86}$$

$$= -\frac{v_c^2}{2\omega_c}\frac{\partial_x B_y}{B_y}\hat{\boldsymbol{z}} \tag{87}$$

$$= -\frac{v_c^2}{2\omega_c}\frac{\lambda^2}{x(x^2+\lambda^2)}\hat{\boldsymbol{z}}. \tag{88}$$

This means that at $x > 0$ region, the orbit of the proton is a cycltron motion in the x-z plane with a drift velocity along the minus z axis.

(c)

For the region of $x < \lambda$ and near the $x \sim 0$ axis, we use the conservation of energy. First, note that a vector potential:

$$\vec{\boldsymbol{A}} = -\sqrt{x^2+\lambda^2}B_0\hat{\boldsymbol{z}} \tag{89}$$

would reproduce the B-field, recalling $\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}$. Then the conservation of energy dictates:

$$\frac{p_x^2}{2m} + \frac{(p_z + qB_0\sqrt{x^2+\lambda^2})^2}{2m} = U. \tag{90}$$

If the component p_z is approximately constant (note according to the equations of motion this is impossible), and we view the effective potential in the x direction as $\frac{(p_z+qB_0\sqrt{x^2+\lambda^2})^2}{2m}$, then we observe: $p_z > 0$ gives a stable minimum at $x = 0$, and thus meaning that the proton would oscillate around $x \sim 0$; $p_z < 0$ gives a "Mexican hat potential", and thus meaning that the proton, given small enough p_x, would oscillate separately in the two valleys. Figure 4 gives the two potentials in the x direction with different signs of p_z. The proton would go along the "snake" orbits by the above analysis.

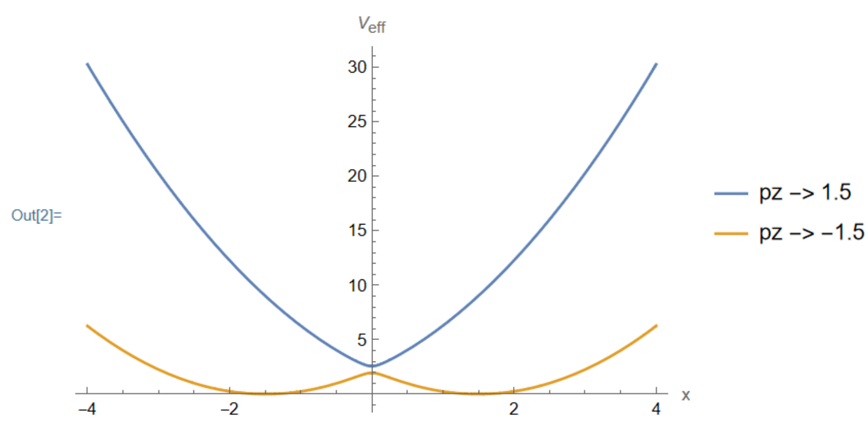

Figure 4: The effective potential for the x axis, with constants $q = B_0 = m = 1$, and $\lambda = 0.1$.

(d)

To get a relation on the critical point where the proton starts to oscillate separately in the $x > 0$ and $x < 0$ regions, we need to study further the equations of motion:

$$\dot{p_x} = -qv_z B_y = -q\dot{z}\frac{B_0 x}{\sqrt{x^2+\lambda^2}} \tag{91}$$

$$\dot{p_z} = qv_x B_y = q\dot{x}\frac{B_0 x}{\sqrt{x^2+\lambda^2}}. \tag{92}$$

Note that the second equation is integrable:

$$m\dot{z} = \int q\dot{x}\frac{B_0 x}{\sqrt{x^2+\lambda^2}} = p_{z0} + qB_0\sqrt{x^2+\lambda^2}. \tag{93}$$

Thus:

$$\ddot{x} = -\frac{qB_0}{m^2}\frac{xp_{z0}}{\sqrt{x^2+\lambda^2}} - \frac{q^2B_0^2}{m^2}x \tag{94}$$

$$\approx -\frac{\omega_0}{m}\frac{xp_{z0}}{\lambda} - \omega_0^2 x = -\omega_0 x\left(\frac{p_{z0}}{\lambda m} + \omega_0\right). \tag{95}$$

Now if $p_{z0} < -\lambda m\omega_0$, then the solution in the x direction is an exponential instead of a sine or cosine function. This is the critical condition. See figure 5.

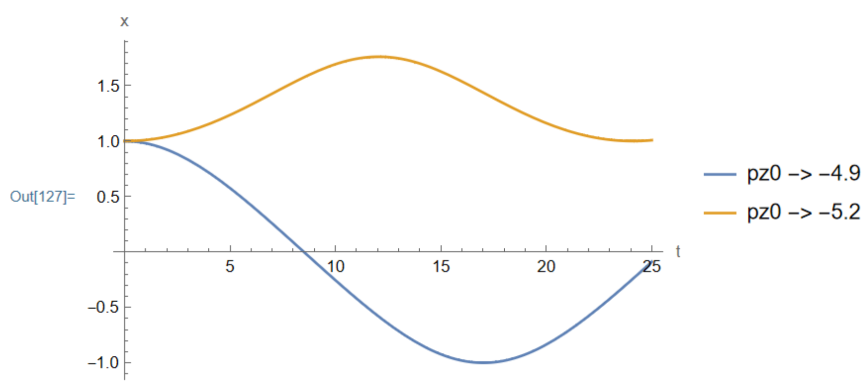

Figure 5: The exact solution for the x direction with different initial p_{z0} values, with $\lambda = 5$, $m = \omega_0 = 1$. It can be observed that with $p_{z0} < -\lambda m\omega_0$ the oscillation is not centered at $x = 0$.