Massachusetts Institute of Technology
Department of Physics
Doctoral General Exam - Part II
Fall 1998
Solution for Electromagnetism 1

Problem 1

An infinitely long wire on the z axis has no current for $t < 0$ and a constant current I for $t > 0$ (in the positive z direction). As in an ordinary wire, where the fixed positive ions cancel the charge of the moving electrons, there is no charge density for all t.

a) Express the current density $\vec{J}(\vec{r}, t)$ using δ-functions and/or step functions.

We use cylindrical coordinates with (z, ρ, ϕ). In this case the current is restricted to the z axis so $\rho = 0$.

$$\vec{J}(\vec{r}, t) = \frac{I}{2\pi\rho}\delta(\rho)\theta(t)\hat{z} \tag{1}$$

It appears that $\vec{J}$ is infinite, and it is. But this is just a by-product of the approximation for an infinitely thin wire carrying the current. The current density goes like I/A where A is the cross-sectional area of the wire which is zero in this case. The 2π ensures that $\int \vec{J} \cdot d\vec{A} = I$.

b) What is the vector potential $\vec{A}(\vec{r}, t)$ for this current? Use the retarded Green's functions and evaluate all integrals. Hints: $\vec{A}(\vec{r}, t) = \vec{A}(\rho, t)$ where ρ is the normal distance to the wire. As the answer is independent of z, consider the observer at $z = 0$.

We can use the formula given in the problem which uses the retarded Green's function to solve for the potential. We just need to know what $|\vec{r} - \vec{r}'|$ is to evaluate the integrals. But since the current is restricted to the z axis and infinite in length, we can set our observer to $z = 0$ for symmetry. Therefore the total distance is $|\vec{r} - \vec{r}'| = \sqrt{\rho^2 + z'^2}$.

$$\begin{aligned}
\vec{A}(\vec{r}, t) &= \frac{1}{c}\int d^3r' \frac{\vec{J}(\vec{r}', t' = t - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r} - \vec{r}'|} \\
&= \frac{I}{2\pi c}\int dz'\rho' d\rho' d\phi' \frac{1}{\rho'\sqrt{\rho^2 + z'^2}}\delta(\rho')\theta(t - \frac{1}{c}\sqrt{\rho^2 + z'^2})\hat{z}
\end{aligned}$$

The radial integration is trivial with the δ-function, and the azimuthal integation simply gives 2π. With the θ-function in our integral, this will limit our integration over the z axis. It means that $-\sqrt{c^2t^2 - \rho^2} < z < \sqrt{c^2t^2 - \rho^2}$. However, we need to restrict our solution to times after the change has had a chance to propagate to the observer, so there will still be $\theta(t - \frac{\rho}{c})$.

$$\begin{aligned}
\vec{A}(\vec{r}, t) &= \frac{I}{c}\int_{-\sqrt{c^2t^2-\rho^2}}^{\sqrt{c^2t^2-\rho^2}} \frac{dz'}{\sqrt{\rho^2 + z'^2}}\theta(t - \frac{\rho}{c})\hat{z} \\
&= \frac{2I}{c}\sinh^{-1}\left(\sqrt{\frac{c^2t^2}{\rho^2} - 1}\right)\theta(t - \frac{\rho}{c})\hat{z}
\end{aligned} \tag{2}$$

We can see that this is independent of z and ϕ, as we expected.

c) Find the magnetic field $\vec{B}(\vec{r},t)$ corresponding to $\vec{A}(\vec{r},t)$. Plot your answer in two ways; in the first show the magnetic field at a fixed point as a function of time, and in the second show the magnetic field at a fixed time as a function of the radial distance to the wire.

The magnetic field is the curl of the vector potential, but since our potential only depends on ρ and t and points in the z direction, there is only one derivative we need to calculate.

$$\begin{aligned}
\vec{B}(\vec{r},t) &= \nabla \times \vec{A} = -\frac{\partial A_z}{\partial \rho}\hat{\phi} \\
&= -\frac{2I}{c}\left[\frac{1}{\sqrt{1+(\frac{c^2t^2}{\rho^2}-1)}}\frac{1}{2\sqrt{\frac{c^2t^2}{\rho^2}-1}}\frac{-2c^2t^2}{\rho^3}\theta(t-\frac{\rho}{c}) + \sinh^{-1}\left(\sqrt{\frac{c^2t^2}{\rho^2}-1}\right)\delta(t-\frac{\rho}{c})\frac{-1}{c}\right]\hat{\phi} \\
&= \frac{2I}{c}\left[\frac{ct}{\rho^2}\frac{1}{\sqrt{\frac{c^2t^2}{\rho^2}-1}}\theta(t-\frac{\rho}{c}) + \frac{1}{c}\sinh^{-1}\left(\sqrt{\frac{c^2t^2}{\rho^2}-1}\right)\delta(t-\frac{\rho}{c})\right]\hat{\phi} \qquad (3)
\end{aligned}$$

Again, this is independent of z and ϕ. We see that these are loops around the wire, which is the usual result for a long current-carrying wire. Also, in the steady-state limit $t \gg \rho/c$, this approaches the traditional result $\vec{B} = \frac{2I}{c\rho}$. It should be noted that the term with the δ-function will not contribute to the solution because the $\sinh^{-1}$ factor is zero when it's own argument is non-zero, but there is a disconitnuity in the solution at this point. Now let's look at the plots.

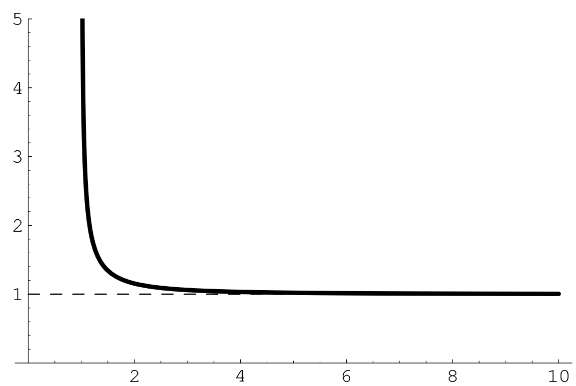

Figure 1: A plot at a fixed distance from the wire. Time is plotted in units of $\frac{\rho}{c}$. The magnetic field starts at zero, undergoes an initial spike, then quickly goes to its steady-state value $\frac{2I}{c\rho}$, indicated with the dashed line.

d) Find the electric field $\vec{E}(\vec{r},t)$. What is its behavior at large times?

There is no net charge present, so the scalar potential is zero. Then using the formula for the electric field, we find

$$\vec{E}(\vec{r},t) = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

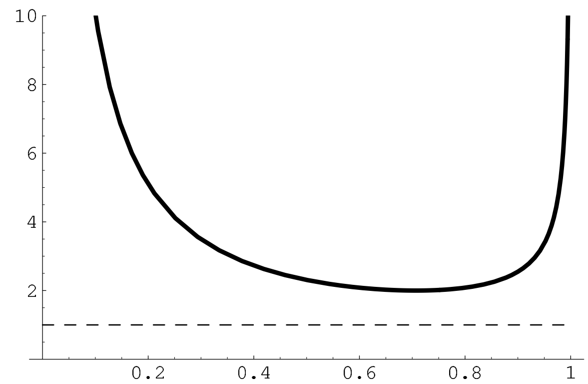

Figure 2: A plot at a fixed time over the distance from the wire. The field is only non-zero for a radial distance less than ct, which corresponds to 1 in this plot. We can see the field is very large near the wire (where it goes like $1/\rho$) and out at the initial spike when the current was turned on. The dashed line indicates the steady-state value of the field.

$$= -\frac{2I}{c^2}\left[\frac{1}{\sqrt{1+(\frac{c^2t^2}{\rho^2}-1)}}\frac{1}{2\sqrt{\frac{c^2t^2}{\rho^2}-1}}\frac{2c^2t}{\rho^2}\theta(t-\frac{\rho}{c})+\sinh^{-1}\left(\sqrt{\frac{c^2t^2}{\rho^2}-1}\right)\delta(t-\frac{\rho}{c})\right]\hat{z}$$

$$= -\frac{2I}{c^2}\left[\frac{c}{\rho}\frac{1}{\sqrt{\frac{c^2t^2}{\rho^2}-1}}\theta(t-\frac{\rho}{c})+\sinh^{-1}\left(\sqrt{\frac{c^2t^2}{\rho^2}-1}\right)\delta(t-\frac{\rho}{c})\right]\hat{z}. \quad (4)$$

The electric field is pointing in the $-z$ direction. Looking at the Poynting vector, $\vec{E}\times\vec{B}$, this points in the $+\hat{\rho}$ direction, so the wire is radiating outward as we expect. Again, the δ-function term will not contribute. For large times $t \gg \rho/c$, the electric field will die off.

$$\vec{E}(t\to\infty)\sim -\frac{2I}{c^2}\frac{1}{t}\hat{z}\longrightarrow 0. \quad (5)$$

In conjunction with this, the magnetic field will approach its steady-state value.