Classical Mechanics Problem 1: Central Potential
Solution

a) Integrals of motion for a central potential $V(r)$:

Angular Momentum $\quad L = rv_t = r^2\dot{\phi}$
Energy per unit mass $\quad E = \frac{1}{2}\left(\dot{r}^2 + v_t^2\right) + V(r) = \frac{1}{2}\dot{r}^2 + V_{\rm eff}(r)$

where v_t is the tangential velocity and $V_{\rm eff}$ is defined as

$$V_{\rm eff}(r) = V(r) + \frac{L^2}{2r^2}$$

If the orbit is circular, the distance of the test body from the origin is invariant: $\dot{r} = 0$, which implies that the body is always at the equilibrium-distance:

$$\frac{dV_{\rm eff}}{dr} = 0 \quad \Rightarrow \quad \frac{dV}{dr} = \frac{L^2}{r^3} = \frac{v_t^2}{r} = r\dot{\phi}^2$$

then

$$\dot{\phi} = \omega_\phi = \frac{L}{r^2} = \left(\frac{1}{r}\frac{dV}{dr}\right)^{1/2}$$

so for the period we get

$$\boxed{T_\phi = \frac{2\pi}{\omega_\phi} = 2\pi\left(\frac{1}{r}\frac{dV}{dr}\right)^{-1/2}}$$

b) Write the orbit as in the statement of the problem:

$$r(t) = r_0 + \epsilon(t) \quad \text{with} \quad \frac{dV_{\rm eff}}{dr}(r_0) = 0 \quad \text{and} \quad \epsilon^2 \ll r_0^2.$$

The energy per unit mass is now $E = \frac{1}{2}\dot{\epsilon}^2 + V_{\rm eff}(r_0 + \epsilon)$, and since ϵ is small we may Taylor-expand the potential as

$$V_{\rm eff}(r_0 + \epsilon) = V_{\rm eff}(r_0) + \underbrace{\frac{dV_{\rm eff}}{dr}(r_0)\epsilon}_{=0} + \frac{1}{2}\frac{d^2V_{\rm eff}}{dr^2}(r_0)\epsilon^2 + \mathcal{O}(\epsilon^3)$$

so then

$$E - V_{\rm eff}(r_0) = \frac{1}{2}\dot{\epsilon}^2 + \frac{1}{2}\frac{d^2V_{\rm eff}}{dr^2}(r_0)\epsilon^2 + \mathcal{O}(\epsilon^3) = \text{const.}$$

In the above equation we readily recognize the equation of the simple harmonic oscillator with

$$\boxed{\omega_r = \left(\frac{d^2V_{\rm eff}}{dr^2}\right)^{1/2}_{r=r_0}}$$

and its general solution is

$$\boxed{\epsilon(t) = \frac{\sqrt{E - V_{\text{eff}}(r_0)}}{\omega_r} \cos\left[\omega_r (t - t_0)\right]}$$

where t_0 is an arbitrary constant.

Now return to writing ω_r in terms of $V(r)$ instead of $V_{\text{eff}}(r)$.

$$\omega_r^2 = \frac{d^2 V_{\text{eff}}}{dr^2} = \frac{d^2 V}{dr^2} + \frac{3L^2}{r^4} = \frac{d^2 V}{dr^2} + 3\omega_\phi^2 = \frac{d^2 V}{dr^2} + \frac{3}{r}\frac{dV}{dr}$$

$$\boxed{\omega_r = \left(\frac{d^2 V}{dr^2} + \frac{3}{r}\frac{dV}{dr}\right)^{1/2}_{r=r_0} = \left[\frac{1}{r^3}\frac{d}{dr}\left(r^3 \frac{dV}{dr}\right)\right]^{1/2}_{r=r_0}}$$

And the radial period is

$$\boxed{T_r = \frac{2\pi}{\omega_r}}$$

c) Stability is determined by the sign of ω_r^2. For stability: $\omega_r^2 > 0$, so

$$\frac{1}{r^3}\frac{d}{dr}\left(r^3 \frac{dV}{dr}\right) > 0$$

for the Yukawa-potential

$$V(r) = -\frac{GM}{r} e^{-kr}$$

so the condition is

$$\frac{1}{r^3}\frac{d}{dr}\left(r^3 \frac{dV}{dr}\right) = \frac{GM}{r^3} e^{-kr}\left[1 + kr - (kr)^2\right] > 0 \quad \Rightarrow \quad \left[1 + kr - (kr)^2\right] > 0$$

$$\left[1 + kr - (kr)^2\right] = \left(\frac{\sqrt{5} - 1}{2} + kr\right)\left(\frac{\sqrt{5} + 1}{2} - kr\right) > 0$$

which is satisfied only if

$$kr < \left(\frac{\sqrt{5} + 1}{2}\right)$$

Therefore circular orbits are unstable for

$$kr > \left(\frac{\sqrt{5} + 1}{2}\right)$$

■

d) The outermost stable circular orbit is at

$$r_0 = \left(\frac{\sqrt{5}+1}{2k}\right)$$

its energy per unit mass is

$$E = V(r_0) + \frac{1}{2}(r_0\omega_\phi)^2 = V(r_0) + \frac{1}{2}\left(r\frac{dV}{dr}\right)_{r=r_0}$$

$$\frac{1}{2}\frac{GM}{r_0}e^{-kr_0}(kr_0 - 1) = \frac{GM}{r_0}e^{-kr_0}\left(\frac{\sqrt{5}-1}{4}\right) > 0$$

If r_0 is decreased only slightly, $E > 0$ still and the orbit is absolutely stable ■
The effective potential for the Yukawa-potential has the form shown in Figure 1.

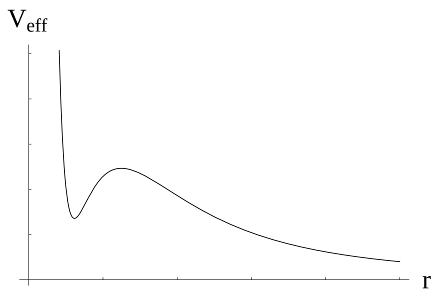
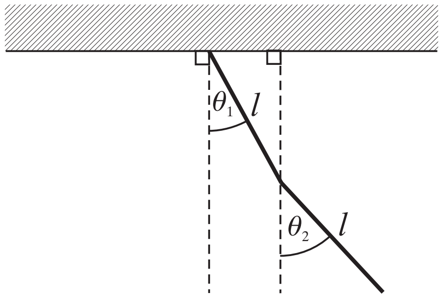


Figure 1: Effective potential against distance from the origin

Classical Mechanics Problem 2: Planar Double Pendulum
Solution



a) $L = T - V$

The moment of inertia for a uniform rod of length l and mass m is

$$I = \frac{1}{3}ml^2 \quad \text{about one of the ends}$$

and

$$I_c = \frac{1}{12}ml^2 \quad \text{about the rod's center}$$

The kinetic energy term we can decompose into three parts:

$$T = T_1 + T_{2,rot} + T_{2,trans}$$

where T_1 is the kinetic energy of the first rod, $T_{2,trans}$ is the translational energy of the center of mass of the second rod and $T_{2,rot}$ is its rotational energy about its center of mass. Then

$$T_1 = \frac{1}{6}ml^2\dot{\theta}_1^2$$

$$T_{2,rot} = \frac{1}{24}ml^2\dot{\theta}_2^2$$

and

$$T_{2,trans} = \frac{1}{2}m\left(\dot{x}_c^2 + \dot{y}_c^2\right)$$

where x_c and y_c are the coordinates of the second rod's center of mass, so

$$x_c = l\sin\theta_1 + \frac{l}{2}\sin\theta_2$$

$$y_c = -l\cos\theta_1 - \frac{l}{2}\cos\theta_2$$

from which

$$\dot{x}_c^2 + \dot{y}_c^2 = l^2\left[\dot{\theta}_1^2 + \frac{1}{4}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2\left(\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2\right)\right]$$

The potential energies are simply $V_i = mgy_{c,i}$, where $y_{c,i}$ are the vertical coordinates of the rods' centers of mass. Since both rods are uniform, $y_{c,i}$ are simply the coordinates of the centers. Thus,

$$V_1 = -mg\frac{l}{2}\cos\theta_1; \quad V_2 = -mg\left(l\cos\theta_1 + \frac{l}{2}\cos\theta_2\right)$$

The full Lagrangian is then

$$L = T_1 + T_{2,rot} + T_{2,trans} - V_1 - V_2$$
$$\boxed{= ml^2\left[\frac{2}{3}\dot{\theta}_1^2 + \frac{1}{6}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right] + mgl\left[\frac{3}{2}\cos\theta_1 + \frac{1}{2}\cos\theta_2\right]}$$

b) Expand the Langrangian from part **a)** for small angles. The only function we have to deal with is

$$\cos\theta = 1 - \frac{1}{2}\theta^2 + \mathcal{O}(\theta^4)$$

Since we are going to look for normal modes with $\theta_j = \hat{\theta}_j \exp(i\omega t)$, where the $\hat{\theta}_j \ll 1$, we immediately see that in the term $\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)$, the θ-dependence in the cosine can be dropped, because even the first θ-dependent term gives a fourth order correction. Then the approximate Lagrangian is

$$L = ml^2\left[\frac{2}{3}\dot{\theta}_1^2 + \frac{1}{6}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2\right] - mgl\left[\frac{3}{4}\theta_1^2 + \frac{1}{4}\theta_2^2\right] + \text{const.}$$

The Euler-Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}_j} = \frac{\partial L}{\partial\theta_j}$$

so in the specific case:

$$\frac{4}{3}\ddot{\theta}_1 + \frac{1}{2}\ddot{\theta}_2 + \frac{g}{l}\left(\frac{3}{2}\theta_1\right) = 0$$
$$\frac{1}{2}\ddot{\theta}_1 + \frac{1}{3}\ddot{\theta}_2 + \frac{g}{l}\left(\frac{1}{2}\theta_2\right) = 0$$

if we now look for normal modes, as mentioned, the above set of equations takes the form

$$\begin{bmatrix} \left(\frac{4}{3}\omega^2 - \frac{3}{2}\frac{g}{l}\right) & \frac{1}{2}\omega^2 \\ \frac{1}{2}\omega^2 & \left(\frac{1}{3}\omega^2 - \frac{1}{2}\frac{g}{l}\right) \end{bmatrix}\begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix} = 0$$

Non-trivial solutions exist if the determinant of the matrix on the left is zero. Denoting $\omega^2 = \lambda g/l$, we can write this condition as

$$\left(\frac{4}{3}\lambda - \frac{3}{2}\right)\left(\frac{1}{3}\lambda - \frac{1}{2}\right) - \frac{\lambda^2}{4} = 0,$$

that is

$$\frac{7}{36}\lambda^2 - \frac{7}{6}\lambda + \frac{3}{4} = 0$$

whose solutions are

$$\lambda_\pm = 3 \pm \frac{6}{\sqrt{7}},$$

so finally

$$\boxed{\omega_\pm = \left[\left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}\right]^{1/2}}$$

c) To sketch the eigenmodes, find eigenvectors of the matrix in part **b)**.

- $\omega^2 = \lambda_- g/l$ (low-frequency mode)

$$\hat{\theta}_2 = \left(\frac{3}{\lambda} - \frac{8}{3}\right)\hat{\theta}_1 = \frac{1}{3}\left(2\sqrt{7} - 1\right)\hat{\theta}_1$$

$(2\sqrt{7} - 1)/3 > 0$ and real, therefore the two pendula are in phase;

- $\omega^2 = \lambda_+ g/l$ (high-frequency mode)

$$\hat{\theta}_2 = \left(\frac{3}{\lambda} - \frac{8}{3}\right)\hat{\theta}_1 = \frac{1}{3}\left(-2\sqrt{7} - 1\right)\hat{\theta}_1$$

$(-2\sqrt{7} - 1)/3 < 0$ and real, therefore the two pendula are perfectly out of phase.

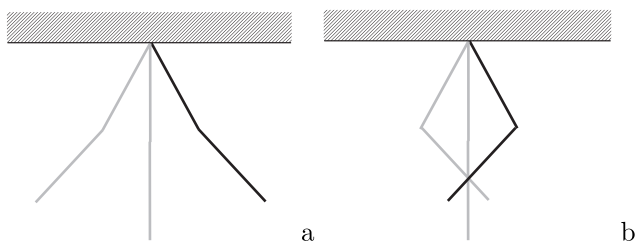


Figure 2: The low- (a) and high-frequency (b) normal modes of the planar double pendulum.

Electromagnetism Problem 1
Solution

a) Normal modes are products of harmonic standing waves in the x, y and z directions. For their frequencies, we have

$$\omega = c\sqrt{k_x^2 + k_y^2 + k_z^2} = c\left[\left(\frac{\pi n_x}{a}\right)^2 + \left(\frac{\pi n_y}{b}\right)^2 + \left(\frac{\pi n_z}{b}\right)^2\right]^{1/2} ; \qquad n_x, n_y, n_z \in \mathbb{Z}^+$$

Since $a > b$, the lowest frequency has $n_x = 1$ and either $n_y = 1, n_z = 0$ or $n_y = 0, n_z = 1$ (note that $n_y = 0$, $n_z = 0$ does not satisfy the boundary condition $E_{\parallel,\text{at wall}} = 0$). Since we are told to pick the mode with $\vec{E} \parallel \hat{y}$, the boundary conditions require

$$\boxed{\vec{E}(r,t) = E_0 \hat{y} \sin\frac{\pi x}{a} \sin\frac{\pi z}{b} \cos\omega t}$$

The magnetic induction we can get from Faraday's Law:

$$\begin{aligned}
\frac{\partial \vec{B}}{\partial t} &= -c\left(\nabla \times \vec{E}\right) = \hat{x}c\frac{\partial E_y}{\partial z} - \hat{z}c\frac{\partial E_y}{\partial x} \\
&= \left(\hat{x}cE_0\frac{\pi}{b}\sin\frac{\pi x}{a}\cos\frac{\pi z}{b} - \hat{z}cE_0\frac{\pi}{a}\cos\frac{\pi x}{a}\sin\frac{\pi z}{b}\right)\cos\omega t
\end{aligned}$$

$$\boxed{\vec{B}(r,t) = -\frac{\pi c E_0}{\omega}\left(\frac{\hat{x}}{b}\sin\frac{\pi x}{a}\cos\frac{\pi z}{b} - \frac{\hat{z}}{a}\cos\frac{\pi x}{a}\sin\frac{\pi z}{b}\right)\sin\omega t}$$

where the frequency ω is (by the argument above)

$$\boxed{\omega = \pi c\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^{1/2}}$$

b) At a boundary of media, the discontinuity in the *normal* component of the electric field is 4π times the surface charge density σ, so

$$\begin{aligned}
E_y(x,0,z) &= 4\pi\sigma \\
\sigma(x,0,z) &= \frac{E_0}{4\pi}\sin\frac{\pi x}{a}\sin\frac{\pi z}{b}\cos\omega t \\
\sigma(x,b,z) &= -\sigma(x,0,z)
\end{aligned}$$

and

$$\sigma(0,y,z) = \sigma(a,y,z) = \sigma(x,y,0) = \sigma(x,y,b) \equiv 0$$

Similarly, at the boundary of media the discontinuity of the *tangential* component of the magnetic field is given by the surface current $\vec{\kappa}$

$$\hat{n} \times \vec{B} = \frac{4\pi}{c}\vec{\kappa}$$

where $\hat{n}$ is a unit vector normal to the surface, so

$$\begin{aligned}
\vec{\kappa}(x,0,z) &= \frac{c^2 E_0}{4\omega}\left(\frac{\hat{z}}{b}\sin\frac{\pi x}{a}\cos\frac{\pi z}{b} + \frac{\hat{x}}{a}\cos\frac{\pi x}{a}\sin\frac{\pi z}{b}\right)\sin\omega t \\
\vec{\kappa}(x,b,z) &= -\vec{\kappa}(x,0,z) \\
\vec{\kappa}(0,y,z) &= -\frac{c^2 E_0}{4\omega}\frac{\hat{y}}{a}\sin\frac{\pi z}{b}\sin\omega t \\
\vec{\kappa}(a,y,z) &= -\vec{\kappa}(0,y,z) \\
\vec{\kappa}(x,y,0) &= -\frac{c^2 E_0}{4\omega}\frac{\hat{y}}{b}\sin\frac{\pi x}{a}\sin\omega t \\
\vec{\kappa}(x,y,b) &= -\vec{\kappa}(x,y,0)
\end{aligned}$$

c) Since there is no charge on the $b \times b$ sides, the force there is purely magnetic and is given by

$$\vec{F}(t) = \frac{1}{2c}\int_{b\times b}\left(\vec{\kappa}\times\vec{B}\right)d^2x$$

$$\begin{aligned}
\vec{F}(x=0,t) &= -\frac{E_0^2 c^2 \pi}{8\omega^2 a^2}\hat{x}\int_0^b dy\int_0^b dz\sin^2\frac{\pi z}{b}\sin^2\omega t \\
&= \boxed{-\hat{x}\left(\frac{c}{4\omega}\frac{b}{a}E_0\sin\omega t\right)^2} \\
\vec{F}(x=a,t) &= -\vec{F}(x=0,t)
\end{aligned}$$

The forces point outwards from the box on both sides (as is indicated by the sign in the equation above).

d) Start with the sides where $y = \text{const}$. The magnetic component of the force can be written as above

$$\begin{aligned}
\vec{F}_{\text{mag}}(y=0,t) &= -\frac{E_0^2 c^2 \pi}{8\omega^2}\hat{y}\int_0^a dx\int_0^b dz\left(\frac{1}{a^2}\cos^2\frac{\pi x}{a}\sin^2\frac{\pi z}{b} + \frac{1}{b^2}\sin^2\frac{\pi x}{a}\cos^2\frac{\pi z}{b}\right)\sin^2\omega t \\
&= -\hat{y}\frac{1}{2}\left(\frac{c}{2\omega}E_0\sin\omega t\right)^2\frac{1}{4\pi}\left(\frac{b}{a}+\frac{a}{b}\right)
\end{aligned}$$

To simplify this result further, use ω from part a)

$$\omega^{-2} = (\pi c)^{-2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^{-1} = ab(\pi c)^{-2}\left(\frac{b}{a}+\frac{a}{b}\right)^{-1}$$

Then

$$\vec{F}_{\text{mag}}(y=0,t) = -\hat{y}\left(\frac{E_0}{4}\sin\omega t\right)^2\frac{ab}{2\pi^3}$$

The electric component of the force can be written as

$$\vec{F}_{\rm el}(y=0,t) = \int \frac{1}{2}\sigma \vec{E}\, d^2x = \hat{y}\frac{E_0^2}{8\pi}\cos^2\omega t \int_0^a dx \int_0^b dz \sin^2\frac{\pi x}{a}\sin^2\frac{\pi z}{b}$$

$$= \hat{y}\frac{1}{2}\left(\frac{E_0}{4}\cos\omega t\right)^2 \frac{ab}{\pi^3}$$

$$\vec{F}_{\rm tot}(y=0,t) = \vec{F}_{\rm el}(y=0,t) + \vec{F}_{\rm mag}(y=0,t)$$

$$= \hat{y}\left(\frac{E_0}{4}\right)^2 \frac{ab}{2\pi^3}\left(\cos^2\omega t - \sin^2\omega t\right)$$

$$= \boxed{\hat{y}\left(\frac{E_0}{4}\right)^2 \frac{ab}{2\pi^3}\cos 2\omega t}$$

and

$$\boxed{\vec{F}_{\rm tot}(y=b,t) = -\vec{F}_{\rm tot}(y=0,t)}$$

There net force on the top and bottom sides oscillates between the inward and outward direction with half the period of the lowest frequency mode. In a time average, therefore, this force cancels.

Next, calculate the force on the sides where $z = \text{const}$. Again, there is no charge, therefore no electric component; the force is purely magnetic

$$\vec{F}(z=0,t) = -\frac{E_0^2 c^2 \pi}{8\omega^2 b^2}\hat{z}\int_0^b dy \int_0^a dx \sin^2\frac{\pi x}{a}\sin^2\omega t$$

$$= \boxed{-\hat{z}\frac{a}{b}\left(\frac{c}{4\omega}E_0 \sin\omega t\right)^2}$$

$$\vec{F}(z=b,t) = -\vec{F}(z=0,t)$$

The magnetic force is pushing the $a \times b$ walls outwards, too (sign!).

e) From the Maxwell stress tensor, the force per unit surface area is

$$\vec{f} = \frac{1}{4\pi}\vec{E}(\vec{E}\cdot\hat{n}) - \frac{E^2}{8\pi}\hat{n} + \frac{1}{4\pi}\vec{B}(\vec{B}\cdot\hat{n}) - \frac{B^2}{8\pi}\hat{n}$$

On the $x = \text{const.}$ walls $\hat{n} = \pm\hat{x}$, $\vec{E} = 0$ and $\vec{B}\cdot\hat{x} = 0$, so

$$\vec{f}(x=\{0,a\},t) = \mp\frac{B^2}{8\pi}\hat{x} = \mp\frac{E_0^2 c^2 \pi}{8\omega^2 a^2}\hat{x}\sin^2\frac{\pi z}{b}\sin^2\omega t$$

which is exactly the integrand from part c).

On the $y = \text{const.}$ walls $\hat{n} = \pm\hat{y}$, $\vec{E}(\vec{E}\cdot\hat{y}) = E^2\hat{y}$ and $\vec{B}\cdot\hat{y} = 0$, so we get

$$\begin{aligned}\vec{f}(y=\{0,b\},t) &= \mp\frac{1}{8\pi}(E^2 - B^2)\hat{y} \\ &= \pm\left[\frac{E_0^2}{8\pi}\cos^2\omega t\sin^2\frac{\pi x}{a}\sin^2\frac{\pi z}{b}\right. \\ &\quad\left.-\frac{E_0^2c^2\pi}{8\omega^2}\left(\frac{1}{a^2}\cos^2\frac{\pi x}{a}\sin^2\frac{\pi z}{b} + \frac{1}{b^2}\sin^2\frac{\pi x}{a}\cos^2\frac{\pi z}{b}\right)\sin^2\omega t\right]\hat{y}\end{aligned}$$

the sum of the first two integrands from part d).
On the $z = \text{const.}$ walls $\hat{n} = \pm\hat{z}$, $(\vec{E}\cdot\hat{z}) = 0$ and $\vec{B}\cdot\hat{z} = 0$, so we get

$$\vec{f}(z=\{0,b\},t) = \mp\frac{B^2}{8\pi}\hat{z} = \mp\frac{E_0^2c^2\pi}{8\omega^2b^2}\hat{z}\sin^2\frac{\pi x}{a}\sin^2\omega t$$

the last integrand from part d).

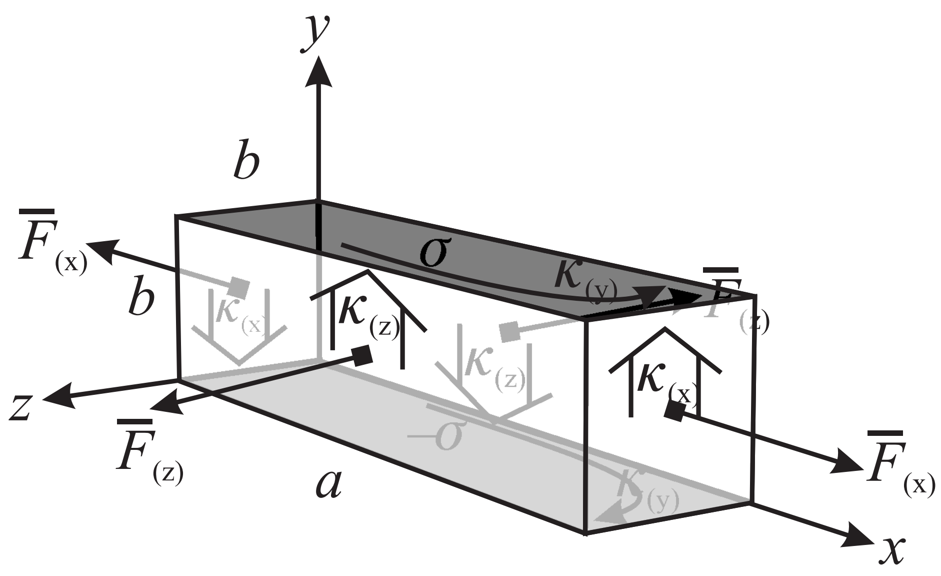


Figure 3: Average total forces, surface charges and surface currents on the cavity.

Electromagnetism Problem 2: Waves in a Dilute Gas
Solution
(see Feynman Lectures on Physics, vol. II, chapter 32)

a) The EM wave is travelling in the $\hat{\mathbf{x}}$ direction; it has a transverse electric field, so assume $\mathbf{E} \times \hat{\mathbf{y}} = 0$. Then the electron in the atom behaves classically as a damped, driven harmonic oscillator

$$m_e \left(\ddot{y} + \gamma \dot{y} + \omega_0^2 y\right) = -qE_0 e^{-i\omega t}$$

with the solution

$$y(t) = \frac{1}{\omega^2 - \omega_0^2 + i\gamma\omega} \frac{qE(t)}{m_e}.$$

For the dipole moment per unit volume:

$$P = n_a(-q)y = \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \frac{n_a q^2 E}{m_e}$$

Therefore the volume polarizability is, according to the definition given,

$$\boxed{\alpha(\omega) = \frac{P}{\epsilon_0 E} = \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \frac{n_a q^2}{\epsilon_0 m_e}}$$

(A quantum mechanical derivation would give this same expression multiplied by the oscillator strength f for the transition.)

b) With no free charges or currents, Maxwell's equations read

$$\nabla \cdot \mathbf{D} = 0; \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0; \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

and $\mathbf{B} = \mu_0 \mathbf{H}$, $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0(1 + \alpha)\mathbf{E}$ for a single frequency ω. This gives us the following wave-equation

$$\frac{\partial^2 \mathbf{D}}{\partial t^2} - \frac{1}{\mu_0 \epsilon_0 (1 + \alpha)} \nabla^2 \mathbf{D} = 0.$$

Now let $\mathbf{D}$ be that of a plane wave: $\mathbf{D} \propto e^{i(kx - \omega t)}$. Then

$$k^2 = \mu_0 \epsilon_0 (1 + \alpha)\omega^2 = (1 + \alpha)\frac{\omega^2}{c^2}$$
$$\Rightarrow \boxed{n(\omega) = \sqrt{1 + \alpha(\omega)}}$$

One can also get this result by using the microscopic $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{P}$ fields:

$$\nabla \cdot \mathbf{E} = -\frac{1}{\epsilon_0} \nabla \cdot \mathbf{P}; \quad c^2 \nabla \times \mathbf{B} = \frac{\partial}{\partial t}\left(\frac{\mathbf{P}}{\epsilon_0} + \mathbf{E}\right)$$

$$\Rightarrow \frac{\partial^2 \mathbf{E}}{\partial t^2} - c^2 \nabla^2 \mathbf{E} = -\frac{1}{\epsilon_0}\frac{\partial \mathbf{P}}{\partial t}$$

also

$$\frac{\partial^2 \mathbf{P}}{\partial t^2} + \gamma \frac{\partial \mathbf{P}}{\partial t} + \omega_0^2 \mathbf{P} = -\frac{n_a q^2}{m_e}\mathbf{E}.$$

Together these give us $k^2 = (1+\alpha)\omega^2/c^2$ for a plane wave, as before. (Note that we are neglecting dipole-dipole interactions in the dilute gas.)

c) We start by noting that according to Fourier-analysis

$$E(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{i(kx-\omega(k)t)}\hat{E}(k)$$

$$\hat{E}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} E(x,0) = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} dx\, e^{-i(k-k_c)x - x^2/(2\sigma^2)}$$
$$= e^{-i(k-k_c)^2\sigma^2/2}$$

Now Taylor-expand $\omega(k)$ about $k = k_c$:

$$\omega(k) = \omega(k_c) + \left(\frac{d\omega}{dk}\right)_{k_c}(k-k_c) + \mathcal{O}\{(k-k_c)^2\}$$
$$\equiv k_c v_{ph} + v_g(k-k_c) + \mathcal{O}\{(k-k_c)^2\}$$

where, by definition, v_{ph} and v_g are the phase- and group-velocities, respectively. Now let $K = k - k_c$. Then

$$E(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dK\, e^{ik_c(x-v_{ph}t)+iK(x-v_g t)-K^2\sigma^2/2}$$

$$\boxed{E(x,t) = e^{ik_c(x-v_{ph}t)}\frac{e^{(x-v_g t)^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}} = e^{ik_c(x-v_{ph}t)}N(x-v_g t,\sigma)}$$

d) From part **c)**

$$v_g = \frac{d\omega}{dk} = \left(\frac{dk}{d\omega}\right)^{-1} = \frac{c}{n}\left(1+\frac{d\log n}{d\log\omega}\right)^{-1}.$$

For the dilute gas, $n = \sqrt{1+\alpha} \approx 1 + \alpha/2$, which we will write as $n = n_r + i n_i$ (for α is complex)

$$n_r \approx 1 + \frac{n_a q^2}{2\epsilon_0 m_e}\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

$$n_i \approx \frac{n_a q^2}{2\epsilon_0 m_e}\frac{\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

Here the real part n_r of the index of refraction determines the dispersion, and the imaginary part n_i determines the absorption/gain coefficient. At $\omega = \omega_0$:

$$n_r = 1 \quad \text{and} \quad \frac{d \log n_r}{d \log \omega} = -\frac{n_a q^2}{2\epsilon_0 m_e \gamma^2}$$

$$\boxed{v_g = \left(1 - \frac{n_a q^2}{2\epsilon_0 m_e \gamma^2}\right)^{-1} c \approx \left(1 + \frac{n_a q^2}{2\epsilon_0 m_e \gamma^2}\right) c}$$

Note that $v_g > c$ at $\omega = \omega_0$. This is called anomalous dispersion. It does *not* violate causality because *signals* (information) cannot travel faster than the minimum of (v_{ph}, v_g), and now $v_{ph} = c$ (since $n_r = 1$). Also, the waves are damped by the electronic resonance maximally at $\omega = \omega_0$.

Quantum Mechanics Problem 1
Solution

1. The ground state will have no nodes, so we can pick the even part of the general solution of the free Schrödinger equation inside the well. Outside the well, square-integrability demands the solutions to vanish at infinity. The wave-function for the ground state is then

$$|x| < w \qquad\qquad |x| > w$$
$$\psi(x) = \cos kx \qquad\qquad \psi(x) = Ae^{-\alpha|x|}$$

Both the wave-function and its derivative has to be continuous at the boundaries of the well:

$$\psi: \qquad \cos kw = Ae^{-\alpha w}$$
$$\frac{d\psi}{dx}: \qquad -k \sin kw = -\alpha A e^{-\alpha w}$$
$$\Rightarrow \qquad k \tan kw = \alpha$$

Directly from Schrödinger's equation:

$$|x| < w \qquad\qquad |x| > w$$
$$E = \frac{\hbar^2 k^2}{2m} \qquad\qquad E = -\frac{\hbar^2 \alpha^2}{2m} + V_0$$

$$\Rightarrow \qquad \frac{\hbar^2 k^2}{2m} = -\frac{\hbar^2 \alpha^2}{2m} + V_0$$

From which we get the transcendental equation:

$$\boxed{k \tan kw = \left[\frac{2mV_0}{\hbar^2} - k^2\right]^{1/2}}$$

Let k^* denote the positive root of the equation above, and introduce the following notation:

$$k_c = \frac{\sqrt{2mV_0}}{\hbar} \qquad \text{and} \qquad k_{max} = \frac{\pi}{2w}.$$

Clearly, the LHS of the equation diverges at k_{max}; and the RHS describes a circle with radius k_c, as shown in Fig. 4.

For the energy we have

$$\boxed{E = \frac{\hbar^2 {k^*}^2}{2m}}$$

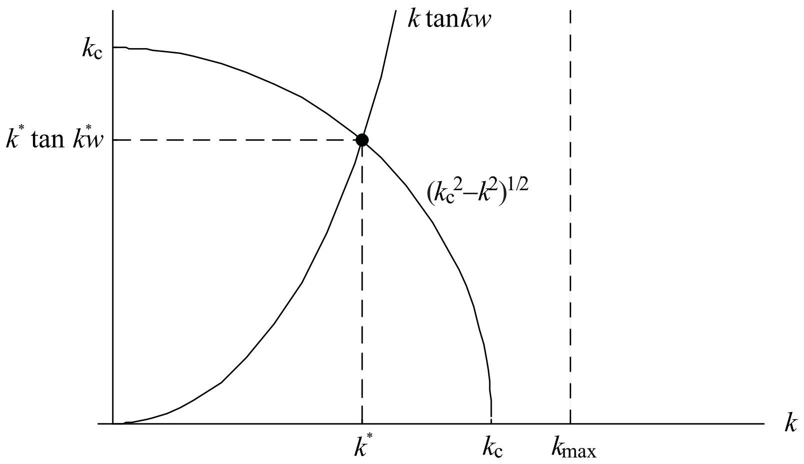


Figure 4: Graphical representation of the solution of the transcendental equation

2. Write the result of part 1 in the non-dimensional form:

$$kw\tan kw = \left[\frac{2mw^2V_0}{\hbar^2} - (kw)^2\right]^{1/2}$$

According to the condition given in the statement of the problem, the radius of the circle on the RHS (that in Fig. 4) goes to infinity, therefore

$$k \to k_{\text{max}}$$

and

$$\boxed{E \to \frac{\hbar^2 k_{\text{max}}^2}{2m} = \frac{\hbar^2\pi^2}{8mw^2}}$$

3. The potential barrier on the low-potential side of the well, denoted a in the figure, will be finite (for any E), so the particle will eventually escape by the tunnel-effect.

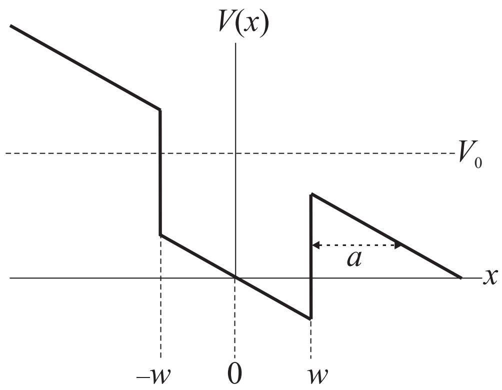
4. $\Delta E = 0$, because the perturbation is odd (and therefore its integral with the square of the ground-state wave-function vanishes).

5.

$$F = \frac{1}{\hbar}\int_w^a dx\,\sqrt{2m(V(x)-B)}$$

$$V(x) = V_0 - e\mathcal{E}x$$

$$B = V_0 - e\mathcal{E}a \quad \Rightarrow \quad a = \frac{V_0 - B}{e\mathcal{E}}$$



$$
\begin{aligned}
F &= \frac{\sqrt{2m}}{\hbar} \int_w^a dx \sqrt{(V_0 - B) - e\mathcal{E}x} \\
&= \frac{\sqrt{2m}}{\hbar} \frac{2}{3} \left(-\frac{1}{e\mathcal{E}}\right) (V_0 - B - e\mathcal{E}x)^{3/2} \Big|_w^a \\
&= \boxed{\frac{\sqrt{2m}}{\hbar} \frac{2}{3} \frac{1}{e\mathcal{E}} (V_0 - B - e\mathcal{E}w)^{3/2}}
\end{aligned}
$$

6. Write the energy of the particle as

$$B = \frac{1}{2}mv^2.$$

Then

$$v^2 = \frac{2B}{m}.$$

The time it takes for the particle to bounce back and forth once is

$$T = \frac{4w}{v},$$

so it hits the right wall with frequency

$$\nu = \frac{v}{4w}$$

$$\Rightarrow \qquad \frac{\text{Probability to escape}}{\text{unit time}} = \frac{v}{4w} e^{-2F}$$

$$\Rightarrow \qquad \text{Lifetime} \sim \frac{4w}{v} e^{2F}$$

Quantum Mechanics Problem 2
Solution

1. Drop the t-label for simplicity. Then we have

$$H = B \begin{bmatrix} \cos\theta & \sin\theta e^{-i\omega t} \\ \sin\theta e^{i\omega t} & -\cos\theta \end{bmatrix},$$

and for the eigenvectors solve

$$\begin{bmatrix} \cos\theta & \sin\theta e^{-i\omega t} \\ \sin\theta e^{i\omega t} & -\cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \pm \begin{bmatrix} x \\ y \end{bmatrix}$$

with the normalization condition $|x|^2 + |y|^2 = 1$. From the vector-equation

$$y = e^{i\omega t} \frac{\sin\theta}{\cos\theta \pm 1} x$$

and with the normalization, we end up with

$$|+\rangle = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) e^{i\omega t} \end{bmatrix} \qquad |-\rangle = \begin{bmatrix} \sin(\theta/2) \\ -\cos(\theta/2) e^{i\omega t} \end{bmatrix}$$

2. Decompose the state-vector as

$$|\psi\rangle = c_+|+\rangle + c_-|-\rangle$$

and write Schrödinger's equation in terms of these vectors:

$$i\hbar \frac{d}{dt}|\psi\rangle = H|\psi\rangle$$

$$i\hbar \left[\dot{c}_+|+\rangle + c_+ \frac{d}{dt}|+\rangle + \dot{c}_-|-\rangle + c_- \frac{d}{dt}|-\rangle\right] = B\left[c_+|+\rangle - c_-|-\rangle\right]$$

or in the $(|+\rangle, |-\rangle)$ basis

$$i\hbar \frac{d}{dt} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} B - i\hbar\langle +|\frac{d}{dt}|+\rangle & -i\hbar\langle +|\frac{d}{dt}|-\rangle \\ -i\hbar\langle -|\frac{d}{dt}|+\rangle & -B - i\hbar\langle -|\frac{d}{dt}|-\rangle \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix}$$

which, with the given concrete form of the vectors, is

$$i\hbar \frac{d}{dt} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} B + \hbar\omega \sin^2(\theta/2) & -\hbar\omega \cos(\theta/2)\sin(\theta/2) \\ -\hbar\omega \sin(\theta/2)\cos(\theta/2) & -B + \hbar\omega\cos^2(\theta/2) \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix}.$$

Now use the identities

$$\sin^2(\theta/2) = \frac{1}{2}(1 - \cos\theta)$$

$$\cos^2(\theta/2) = \frac{1}{2}(1 + \cos\theta)$$

$$\sin(\theta/2)\cos(\theta/2) = \frac{1}{2}\sin\theta$$

to get

$$i\hbar\frac{d}{dt}\left[\begin{array}{c} c_+ \\ c_- \end{array}\right] = \frac{1}{2}\left[\begin{array}{cc} 2B - \hbar\omega\cos\theta & -\hbar\omega\sin\theta \\ -\hbar\omega\sin\theta & -2B + \hbar\omega\cos\theta \end{array}\right]\left[\begin{array}{c} c_+ \\ c_- \end{array}\right].$$

Note that in the last equation we dropped the part of the Hamiltonian that was proportional to the identity, since that gives only a time dependent phase that is identical for the coefficients c_-, c_+. This we can rewrite in the form:

$$i\hbar\frac{d}{dt}\left[\begin{array}{c} c_+ \\ c_- \end{array}\right] = \left[\begin{array}{cc} D_z & D_x \\ D_x & -D_z \end{array}\right]\left[\begin{array}{c} c_+ \\ c_- \end{array}\right].$$

with the solution

$$\begin{aligned}\left[\begin{array}{c} c_+ \\ c_- \end{array}\right] &= \exp\left\{\frac{-i}{\hbar}\left[\begin{array}{cc} D_z & D_x \\ D_x & -D_z \end{array}\right]\right\}\left[\begin{array}{c} 1 \\ 0 \end{array}\right] \\ &= \cos\left(\frac{|\vec{D}|t}{\hbar}\right) - i\hat{D}\cdot\vec{\sigma}\sin\left(\frac{|\vec{D}|t}{\hbar}\right).\end{aligned}$$

And so

$$c_+ = \cos\left(\frac{|\vec{D}|t}{\hbar}\right) - i\frac{D_z}{|\vec{D}|}\sin\left(\frac{|\vec{D}|t}{\hbar}\right)$$

$$\boxed{|c_+|^2 = \cos^2\left(\frac{|\vec{D}|t}{\hbar}\right) - i\frac{D_z^2}{D^2}\sin^2\left(\frac{|\vec{D}|t}{\hbar}\right)}$$

3. For $B \gg \hbar\omega$, $D_z \to D$, so

$$|c_+|^2 \to 1$$

(Adiabatic theorem)

Statistical Mechanics and Thermodynamics Problem 1
Thermodynamics of a Non-Interacting Bose Gas
Solution

a)

$$n_p = \frac{1}{e^{\beta(E_p-\mu)}-1} \qquad\qquad E_p = \frac{p^2}{2m}$$

At and below T_{BEC} $\mu = 0$. At *exactly* T_{BEC}, there are no atoms in the condensate and

$$N = \frac{V}{2\pi\hbar^3}\int \frac{d^3p}{e^{\beta p^2/(2m)}-1} = (2\pi\hbar)^{-3}\,V\left(\frac{2m}{\beta}\right)^{3/2} 4\pi \underbrace{\int_0^\infty \frac{x^2dx}{e^{x^2}-1}}_{=I_1}$$

$$n = \frac{1}{2\pi^2}\left(\frac{2mkT}{\hbar^2}\right)^{3/2} I_1$$

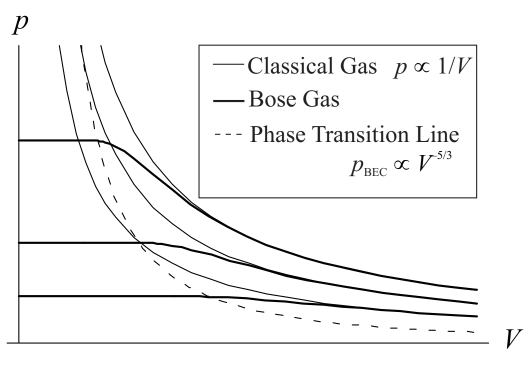
$$\boxed{kT_{\text{BEC}} = \left(\frac{2\pi^2 n}{I_1}\right)^{2/3}\frac{\hbar^2}{2m}}$$

(2 points)

b) The above integral with $\mu = 0$ also applies below T_{BEC}, but it then gives the number of *non-condensed* atoms. So on an isotherm below V_{critical}

- $N_{\text{non-condensed}}$ is constant
- T is constant

$\Rightarrow$ p is constant
(think of the kinetic origin of pressure)



(2 points)

c)

$$U = \frac{V}{2\pi\hbar^3}\int \frac{p^2}{2m}\frac{d^3p}{e^{\beta p^2/(2m)}-1} = (2\pi\hbar)^{-3}\,V\left(\frac{2m}{\beta}\right)^{5/2}\frac{4\pi}{2m}\underbrace{\int_0^\infty \frac{x^4dx}{e^{x^2}-1}}_{=I_2}$$

$$= N_c\frac{I_2}{I_1}\left(\frac{2m}{\beta}\right)\frac{1}{2m} = N_c\frac{I_2}{I_1}kT \propto T^{5/2}$$

$$c_v = \frac{5}{2}\frac{I_2}{I_1}N_ck = \boxed{\frac{V}{2\pi^2}\left(\frac{2mkT}{\hbar^2}\right)^{3/2}k\left(\frac{5}{2}I_2\right)\propto T^{3/2}}$$

(2 points)

d) From the reversibility of the Carnot-cycle:

$$dS_1 = -dS_2 \quad \text{for 1 cycle}$$

$$\Delta S_1 = -\Delta S_2 \quad \text{for the entire process}$$

$$dU = TdS - \underbrace{pdV}_{=0} \quad \Rightarrow \quad T\frac{\partial S}{\partial T}\bigg|_V = \frac{\partial U}{\partial T}\bigg|_V = c_v = \underbrace{aT^{3/2}}_{\text{from c)}}$$

$$\Rightarrow \frac{dS}{dT} = \frac{c_v}{T}$$

Therefore the entropy transfer in the entire process is

$$\Delta S_i = \int_{T_i}^{T_0}\frac{c_v}{T}dT = a\int_{T_i}^{T_0}T^{1/2}dT = \frac{2}{3}a\left(T_0^{3/2}-T_i^{3/2}\right)$$

$$\Delta S_1 + \Delta S_2 = 0 \quad \Rightarrow \quad \boxed{T_0^{3/2} = \frac{1}{2}\left(T_1^{3/2}+T_2^{3/2}\right)}$$

Heat transferred to F_1 : $$Q_1 = \int_{T_1}^{T_0}TdS = \frac{2}{5}a\left(T_0^{5/2}-T_1^{5/2}\right).$$

Heat transferred from F_2 : $$Q_2 = \int_{T_0}^{T_2}TdS = \frac{2}{5}a\left(T_2^{5/2}-T_0^{5/2}\right).$$

Therefore the total work done by the Carnot-machine is

$$\boxed{W = Q_2 - Q_1 = \frac{2}{5}a\left(T_1^{5/2}+T_2^{5/2}-2T_0^{5/2}\right)}$$

(4 points)

Statistical Mechanics and Thermodynamics Problem 2
Phase Transition in a Superconductor
Solution

a)

$$c_H \equiv \left.\frac{\partial Q}{\partial T}\right|_H = T\left.\frac{\partial S}{\partial T}\right|_H$$

$$dS = \left.\frac{\partial S}{\partial T}\right|_M dT + \left.\frac{\partial S}{\partial M}\right|_T dM$$

$$\left.\frac{\partial S}{\partial T}\right|_H = \left.\frac{\partial S}{\partial T}\right|_M + \left.\frac{\partial S}{\partial M}\right|_T \underbrace{\left.\frac{\partial M}{\partial T}\right|_H}_{=0}$$

where the last term is zero because M is independent of T. Then

$$c_H = T\left.\frac{\partial S}{\partial T}\right|_M = \left.\frac{\partial Q}{\partial T}\right|_M \equiv c_M$$

(2 points)

b) The transition takes place at constant T and H. The thermodynamic function whose variables are T and H is the Gibbs-potential:

$$dG = -SdT - MdH$$

$G_{\text{super}} = G_{\text{normal}}$ at every point on $H_{\text{C}}(T)$, so $dG_{\text{S}} = dG_{\text{N}}$ which we then write as

$$-S_{\text{S}}dT - M_{\text{S}}dH = -S_{\text{N}}dT - \underbrace{M_{\text{N}}}_{=0} dH$$

$$\left.\frac{dH}{dT}\right|_{\substack{\text{trans.}\\ \text{line}}} = \frac{dH_{\text{C}}}{dT} = \frac{S_{\text{N}} - S_{\text{S}}}{M_{\text{S}}} = \boxed{-\frac{4\pi}{VH_{\text{C}}(T)}(S_{\text{N}} - S_{\text{S}})}$$

(3 points)

c) By the third law $S \to 0$ as $T \to 0$. But the figure shows $H_{\text{C}}(T = 0)$ is finite. Therefore

$$\frac{dH_{\text{C}}}{dT} \to 0 \quad \text{as} \quad T \to 0.$$

The transition is second order where $S_{\text{N}} - S_{\text{S}} = 0$, that is, the latent heat equals zero.

$$S_{\text{N}} - S_{\text{S}} = -\frac{V}{4\pi}H_{\text{C}}(T)\frac{dH_{\text{C}}}{dT}$$

- At $T = 0$ the transition is second order because both entropies go to zero.
- At $T = T_C(H = 0)$ the transition is second order since $H_C(T) = 0$ and dH_C/dT is finite.
- At all other temperatures the transition is first order since both $H_C(T)$ and dH_C/dT are finite.

(2 points)

d) Use H and T as variables

$$dS(H,T) = \left.\frac{\partial S}{\partial H}\right|_T dH + \left.\frac{\partial S}{\partial T}\right|_H dT$$

$$\left.\frac{\partial S}{\partial H}\right|_T = \frac{c_H}{T} \qquad \left.\frac{\partial S}{\partial T}\right|_H \underset{\substack{\uparrow \\ \text{Maxwell} \\ \text{relation}}}{=} -\left.\frac{\partial M}{\partial T}\right|_H = 0$$

$$\begin{aligned} S = \int \frac{c_H}{T} dT &= \frac{a}{3} T^3 V && T < T_C \\ &= \frac{b}{3} T^3 V + \gamma T V && T > T_C \\ S_N - S_S &= \left(\frac{b-a}{3}\right) T^3 V + \gamma T V && T = T_C(H = 0) \end{aligned}$$

$$\gamma = \left(\frac{b-a}{3}\right) T_C^2$$

$$\boxed{T_C(H = 0) = \left(\frac{3\gamma}{b-a}\right)^{1/2}}$$

(3 points)