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a) (Please see the figure on page 4).

$$Kinetic\ Energy = \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2$$

$$Potential = \frac{1}{2}T\theta^2 = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2,$$

Since, $\theta \approx \frac{\partial y}{\partial x}$.
Hence the Lagrangian can be written as

$$L = \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2$$

The Euler-Lagrange equation can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial(\partial_t y)}\right) + \frac{d}{dx}\left(\frac{\partial L}{\partial(\partial_x y)}\right) = \frac{\partial L}{\partial y}$$

$$\frac{\partial^2 y}{\partial t^2} - \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2} = 0$$

b)
The force on an element of length dx is given by

$$F = T\left(\theta + \frac{\partial \theta}{\partial x}dx\right) - T\theta = T\frac{\partial \theta}{\partial x}dx$$

where, $\theta \approx \frac{\partial y}{\partial x}$, Hence the force on the element is given by

$$F = T\frac{\partial^2 y}{\partial x^2}dx,$$

The equation of motion is given by,

$$F = ma \Rightarrow T\frac{\partial^2 y}{\partial x^2}dx = \rho dx\frac{\partial^2 y}{\partial t^2}$$

$$\rho\frac{\partial^2 y}{\partial t^2} - T\frac{\partial^2 y}{\partial x^2} = 0$$

c)
$y = A\sin(\omega t)\sin\left(\frac{\pi x}{L}\right)$ is a solution of the above equation if,

$$-\rho A\omega^2\sin(\omega t)\sin\left(\frac{\pi x}{L}\right) + T_0 A\left(\frac{\pi}{L}\right)^2\sin(\omega t)\sin\left(\frac{\pi x}{L}\right) = 0$$

Hence the frequency is given by,

$$\omega = \left(\frac{T_0}{\rho}\right)^{1/2} \frac{\pi}{L}$$

d) The wave equation can be reduced to ordinary differential equations using variable separation technique as follows,

$$\frac{\rho}{T(t)Y}\frac{d^2Y}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2}$$

The equation above holds good only if both sides of the equation are constant, i.e.

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k^2 = \frac{\rho}{T(t)}\frac{d^2Y}{dt^2}$$

$$X = A\exp(ikx) + B\exp(-ikx)$$

(k could be complex and is determined from the boundary conditions)
$X(0) = 0$ and $X(L) = 0$
Hence $k = n\pi/L$, $X = A\sin\left(\frac{n\pi x}{L}\right)$

$$\frac{d^2Y}{dt^2} + \left(\frac{n\pi}{L}\right)^2 \frac{T(t)}{\rho}Y = 0$$

Where,

$$\begin{aligned} T(t) &= T_0 \text{ for } t < 0 \\ &= T_0(1+\varepsilon) \text{ for } 0 < t < t_0 \\ &= T_0 \text{ for } t > t_0 \end{aligned}$$

Or $T(t) = T_0\left(1 + \varepsilon\theta(t)\theta(t_0 - t)\right)$. Hence,
$\frac{d^2Y}{dt^2} + \left(\frac{n\pi}{L}\right)^2 \frac{T_0}{\rho}Y = -\left(\frac{n\pi}{L}\right)^2 \frac{T_0\varepsilon\theta(t)\theta(t_0-t)}{\rho}Y$ (*)
The complete solution is a superposition of all modes and can be written as

$$y = \sum_n Y_n(t)\sin\left(\frac{n\pi x}{L}\right)$$

where $Y_n(t)$ satisfies equation (*). Since the string shape for $t < 0$ is $y = A\sin(\omega t)\,\sin\left(\frac{\pi x}{L}\right)$, the string vibrates only in the first mode i.e., $Y_n = 0$ for $n \geq 2$ and the sum reduces to a single term namely,

$$y = Y_1(t)\sin\left(\frac{\pi x}{L}\right)$$

The solution for $Y1$ can be obtained using perturbation technique as follows:

$$Y_1 = Y_1^{(0)} + \varepsilon Y_1^{(1)} + h.o.t.$$

Substituting the expansion for Y_1in Eq. (*) and equating terms of $O\left(\varepsilon^0\right)$ we find that $Y_1^{(0)}$ is the solution of,

$$\frac{d^2Y_1^{(0)}}{dt^2} + \left(\frac{\pi}{L}\right)^2 \frac{T_0}{\rho} Y_1^{(0)} = 0$$

Since the string shape for $t < 0$ is $y = A\sin(\omega t)\sin\left(\frac{\pi x}{L}\right)$,

$$Y_1^{(0)} = A\sin(\omega t)$$

where $\omega = (\pi/L)\,[T_0/\rho]^{1/2}$ [derived in part (c)]. By equating terms of $O\left(\varepsilon^1\right)$ we find that

$$\frac{d^2Y_1^{(1)}}{dt^2} + \omega^2 Y_1^{(1)} = -T_0\theta(t_0 - t)\theta(t)Y_1^{(0)}$$

The above equation can be solved using Green's function technique as follows: The Green's function satisfies the following

$$\frac{d^2G}{dt^2} + \omega^2 G = \delta(t - t')$$

G also satisfies the following jump conditions:

$$G(t'_+) \;=\; G(t'_-); \quad \frac{dG}{dt}(t'_+) \;=\; \frac{dG}{dt}(t'_-) + 1;$$

The Green's function is given by,

$$\begin{aligned} G(t,t') &= \tfrac{\sin\omega(t-t')}{\omega} \text{ for } t > t' \\ &= 0 \text{ for } t < t' \end{aligned}$$

The solution of $Y_1^{(1)}$ is given by,

$$Y_1^{(1)}(t) = -\frac{T_0}{\rho}\int G(t,t')\theta(t')\theta(t_0 - t')Y_1^{(0)}(t')dt'$$

Since, $t > t_0 \geq t'$, the solution can be written as,

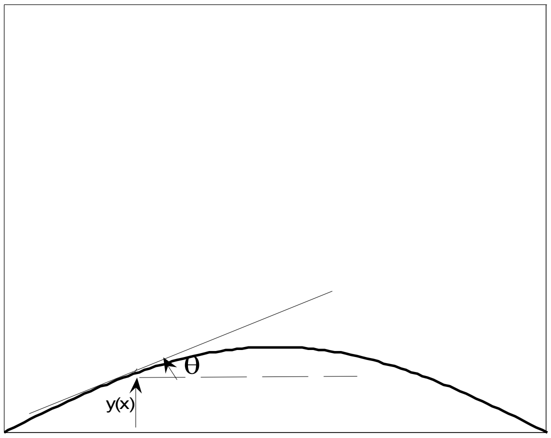
$$Y_1^{(1)}(t) = -\frac{AT_0}{\rho}\int_0^{t_0} dt' \frac{\sin\left[\omega(t - t')\right]}{\omega}\sin\omega t' = -\frac{AT_0}{2\rho\omega}\int_0^{t_0}\left[\cos\left(\omega\left(t - 2t'\right)\right) - \cos\left(\omega t\right)\right]$$

$$Y_1^{(1)}(t) = -\frac{AT_0}{2\rho\omega}\int_0^{t_0}\left[\cos\left(\omega\left(t - 2t'\right)\right) - \cos\left(\omega t\right)\right] = -\frac{AT_0}{2\rho\omega}\left[\frac{-\sin\left(\omega t\right) + \sin\left(\omega\left(2t_0 - t\right)\right)}{2\omega} - t_0\cos\omega t\right]$$

$$= -\frac{AT_0}{2\rho\omega}\left[\frac{\sin\left(\omega t_0\right)\cos\left(\omega\left(t_0 - t\right)\right)}{\omega} - t_0\cos\omega t\right]$$

Hence, for $t > t_0$

$$Y(t) = A\sin\omega t - \frac{AT_0\varepsilon}{2\rho\omega}\left[\frac{\sin\left(\omega t_0\right)\cos\left(\omega\left(t_0 - t\right)\right)}{\omega} - t_0\cos\omega t\right] + O\left(\varepsilon^2\right)$$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