Classical Mechanics Problem 1
Solution

Choose system of coordinates moving with the container where x-axis is along AB line. In that system the angular velocity is given by:

$$\vec{\Omega} = \vec{\omega}_1 + \vec{\omega}_2 = (\omega_2; \omega_1 \cos\omega_2 t; -\omega_1 \sin\omega_2 t)$$

Net force acting on a fluid element of mass m in the non-inertial coordinate system is:

$$\vec{F}_{net} = m\vec{g} + 2m\vec{v} \times \vec{\Omega} + \frac{\partial}{\partial \vec{r}} \frac{m(\vec{\Omega} \times \vec{r})^2}{2} + m(\vec{r} \times \dot{\vec{\Omega}})$$

The hardening resin moves very slowly with $\vec{v} \approx 0$ such that we can ignore Coriolis force

$$2m\vec{v} \times \vec{\Omega} \approx 0$$

The remaining forces need to be averaged over many rotations (time)

$$m\vec{g} = -mg(\cos\omega_1 t;\ \sin\omega_1 t \sin\omega_2 t;\ \sin\omega_1 t \cos\omega_2 t)$$

averaging:

$$\langle m\vec{g} \rangle = \vec{0}$$

For $\omega_1 \neq \omega_2$:

$$\langle m(\vec{r} \times \dot{\vec{\Omega}}) \rangle = m\vec{r} \times \langle \dot{\vec{\Omega}} \rangle = 0$$

We are left with one force that does not average to 0 !

$$(\vec{\Omega} \times \vec{r})^2 = \Omega^2 r^2 - (\vec{\Omega} \cdot \vec{r})^2$$

$$(\vec{\Omega} \times \vec{r})^2 = (\omega_1^2 + \omega_2^2) r^2 - (x\omega_2 + y\omega_1 \cos\omega_2 t - z\omega_1 \sin\omega_2 t)^2$$

Averaging with $\langle \cos^2 \omega t \rangle = \frac{1}{2}$, $\langle \sin^2 \omega t \rangle = \frac{1}{2}$ and $\langle \sin\omega t \cos\omega t \rangle = 0$ we get:

$$\langle (\vec{\Omega} \times \vec{r})^2 \rangle = (\omega_1^2 + \omega_2^2)(x^2 + y^2 + z^2) - (x^2\omega_2^2 - \frac{1}{2}y^2\omega_1^2 - \frac{1}{2}z^2\omega_1^2))$$

This gives the expression for the net force acting on resin element:

$$\vec{F}_{net} = \frac{m}{2}\frac{\partial}{\partial \vec{r}}(\omega_1^2 x^2 + (\frac{\omega_1^2}{2} + \omega_2^2)(y^2 + z^2)) = -\frac{\partial}{\partial \vec{r}}U(\vec{r})$$

The force is due to an effective potential $U(\vec{r})$. The resin will harden with the surface being an equipotential line $U(\vec{r}) = const$. The final shape will be an ellipsoid symmetric around AB axis.

Classical Mechanics Problem 2
Solution

For motion along y-axis one has expression for potential energy:

$$U(y) = k(\sqrt{L^2 + y^2} - L_0)^2$$

Expanding to y^4, ignoring constants etc one gets potential:

$$U(y) = -k(\frac{L_0}{L} - 1)\ y^2 + \frac{kL_0}{4L^3}\ y^4$$

or

$$U(y) = -B\ y^2 + C\ y^4$$

with

$$B = k(\frac{L_0}{L} - 1) \text{ and } C = \frac{kL_0}{4L^3}$$

Equations of motion in terms of y:

$$m\ddot{y} = -\frac{d}{dy}U(y) - my_0\omega^2 \cos\omega t$$

Substituting $s = y - y_0 \cos\omega t$ we get:

$$m\ddot{s} = -\frac{d}{ds}U(s + y_0 \cos\omega t)$$

Since ω is very large the free oscillations of the ball will occur in an effective potential obtained by averaging U(y) over many periods of oscillations of the external force. Expanding and averaging various $\cos\omega t^n$ terms we get:

$$\langle(s + y_0 \cos\omega t)^2\rangle = s^2 + const$$

and

$$\langle(s + y_0 \cos\omega t)^4\rangle = s^4 + \frac{6s^2y_0^2}{2} + const$$

The effective potential will be then:

$$U_{eff}(s) = -Cs^2 + B(s^4 + 3s^2y_0^2) + const$$

or

$$U_{eff}(s) = As^2 + Bs^4$$

where

$$A = 3By_0^2 - C$$

A can be positive or negative, the free oscillatory motion will depend on the sign of A

For $A > 0$ or $T = y_0^2 > \frac{C}{3B}$ the equilibrium point will be at $s_0 = 0$ and the frequency of free small amplitude oscillations will be $\omega = \sqrt{\frac{2A}{m}}$. For $A < 0$ or $T < \frac{C}{3B}$ there will be two equilibria at $s_0 = \pm\sqrt{\frac{-A}{2B}}$ and the frequency of free oscillations will be $\omega = \sqrt{\frac{-A}{m}}$

Part (e):

In case of motion restricted to x-axis the potential is:

$$U(x) = (k/2)((L - L_0) - x)^2 + (k/2)((L - L_0) + x)^2)$$

or

$$U(x) = kx^2 + const$$

Following similar procedure to previoous parts (variable substitution, averaging etc) we get effective potential:

$$U_{eff}(s) = ks^2 + const$$

With $s = x - x_0 \cos \omega t$

This means that the small oscillations will not be affected by the presence of high frequency external force.

The mass will oscillate around $x = 0$ with frequency $\omega = \sqrt{2k/m}$

Electromagnetism Problem 1
Solution

Time varying magnetic field will induce electric field. This in turn will lead to current flowing in resistive medium.

$$\vec{j}(\vec{r}) = \frac{1}{\rho}\vec{E}(\vec{r})$$

In the cylindrical coordinate system the induced electric field will have only tangential component $\hat{\imath}_\theta$ where $\hat{\imath}_r \times \hat{\imath}_\theta = \hat{z}$. Electric field will depend only on the distance from the center of the cylinder r.

From Faraday's Law:

$$2\pi r E_\theta(r) = -\frac{d}{dt}\Phi_{\mathrm{B,enclosed}}$$

$$\Phi_B(r) = \int_0^r 2\pi r' dr' B_z(r')$$

After some elementary calculations and assuming that we have only applied magnetic field (no self-induction approximation) we get circular currents flowing around the axis of the cylinder:

$$\vec{j}(\vec{r}, t) = \frac{B_z r\omega \sin\omega t}{2\rho} \;\; \hat{\imath}_\theta$$

The effect of self inductance will appear due to magnetic field created by the induced currents. Since the induced magnetic field will in turn induce the electric field the current distribution will be modified.

The induced magnetic field can be calculated by dividing cylinder into thin solenoids. The contribution to the magnetic field of a solenoid of radius r and thickness dr will be:

$$d(\Delta\vec{B})(r') = \begin{cases} \mu_0 j(r) dr \hat{z} & \text{if } r' < r; \\ 0, & \text{otherwise.} \end{cases}$$

The induced magnetic field is then:

$$\Delta B(r) = \int_r^R d(\Delta B)(r') dr'$$

$$\Delta\vec{B}(\vec{r}) = \frac{\mu_0 B_z \omega \sin\omega t}{2\rho}\left(\frac{R^2 - r^2}{2}\right)\hat{z}$$

Now we need to calculate again the flux, induced electric field and finally induced current density.

$$\Delta\Phi_B(r) = \int_0^r 2\pi r' dr' \left(\frac{\mu_0 B_z \omega \sin\omega t}{2\rho}\right)\left(\frac{R^2 - r'^2}{2}\right)$$

Final correction to the current density is then:

$$\Delta j(\vec{r}) = -\frac{\mu_0 B_z r \omega^2 \cos\omega t}{4\rho^2}\left(\frac{R^2}{2} - \frac{r^2}{4}\right)$$

The effect of this ω^2 correction will be negligible (to within O(1)) if:

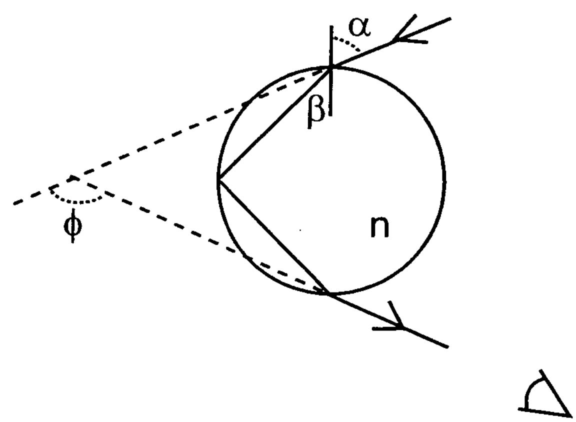
$$\frac{\mu_0 \omega R^2}{2\rho} << 1$$

or

$$\omega << \frac{2\rho}{\mu_0 R^2}$$

Electromagnetism Problem 2
Solution

Set up geometrical optics:



Calculate deflection angle:

$$\phi = 180^0 + 2\alpha - 4\beta$$

From Snell's Law:

$$\beta = \arcsin\left(\frac{\sin\alpha}{n}\right)$$

such that

$$\phi(\alpha) = 180^0 + 2\alpha - 4\arcsin\left(\frac{\sin\alpha}{n}\right)$$

The parallel light shining on the drop within a circle corresponding to angle range α and $\alpha + d\alpha$ will deflect into a cone with opening angle from $\phi(\alpha)$ to $\phi(\alpha + d\alpha)$. The total energy falling on the drop within this angle range is

$$dI = 2\pi I_0 R^2 \sin\alpha \cos\alpha d\alpha$$

Fraction f of this energy will be deflected into the output cone.

The total area of the cone cross section at a distance L from the drop is

$$S_{tot} = 2\pi L^2 \sin\phi d\phi$$

Detector of area S at a distance L from the drop will receive energy:

$$E = \frac{S}{2\pi L^2 \sin\phi d\phi} f I_0 2\pi R^2 \sin\alpha \cos\alpha d\alpha$$

or

$$E = \frac{f I_0 S R^2}{L^2} \frac{\cos\alpha \sin\alpha}{|\frac{d\phi}{d\alpha}| \sin\phi}$$

The rainbow appears since $|\frac{d\phi}{d\alpha}|$ can be 0 at and angle α between 0^0 and 90^0!. E will not be infinite due to light diffraction but it will be very large and certainly larger than the neighboring values of ϕ. The maximum angle depends on n and results in different angular radius for different colors.

Let's calculate the precise location of the extremum angle:

$$\frac{d\phi}{d\alpha} = 2 - \frac{4}{n} \cos\alpha \frac{1}{\sqrt{1 - \frac{\sin^2\alpha}{n^2}}}$$

The derivative is 0 at α_0 such that

$$\cos\alpha_0 = \sqrt{\frac{n^2 - 1}{3}}$$

and

$$\sin\alpha_0 = \sqrt{\frac{4 - n^2}{3}}$$

The angular radius $\chi = 180^0 - \phi$ is then:

$$\chi = 4 \arcsin\sqrt{\frac{4 - n^2}{3n^2}} - 2 \arcsin\sqrt{\frac{4 - n^2}{3}}$$

For the refraction coefficients given in the problem we get:

$$\chi_{red} = 26.66^0$$

and

$$\chi_{violet} = 25.67^0$$

Statistical Mechanics Problem 1
Solution

(a) Up to a normalization factor A,

$$P(q,z) = Ae^{-H(q,z)/kT}$$
$$= Ae^{-\left[\frac{q^2}{2m}+mgz\right]/kT} ,$$

where k is the Boltzmann constant. Then

$$N = \int_0^\infty \mathrm{d}z \int_{-\infty}^\infty \mathrm{d}q\, P(q,z)$$
$$= A\int_0^\infty \mathrm{d}z\, e^{-mgz/kT} \int_{-\infty}^\infty \mathrm{d}q\, e^{-q^2/(2mkT)}$$
$$= A\left(\frac{kT}{mg}\right)\sqrt{2\pi mkT}\,.$$

So

$$A = \frac{N}{\sqrt{2\pi mkT}}\frac{mg}{kT} ,$$

and then

$$\boxed{P(q,z) = \frac{N}{\sqrt{2\pi mkT}}\frac{mg}{kT}e^{-\left[\frac{q^2}{2m}+mgz\right]/kT} .}$$

(b) The magnitude of $|q|$ at a given value of z is determined by conservation of energy:

$$\frac{q^2}{2m} + mgz = mgh \quad\Longrightarrow\quad q^2 = 2m^2g(h-z)\ .$$

Furthermore, the probability of finding the particle in any interval $\mathrm{d}z$ is proportional to the amount of time the particle spends in that interval, which is inversely proportional to the speed at that location. So, up to a normalization constant B,

$$P_1(q,z|h) = \frac{B}{\sqrt{h-z}}\left[\delta\left(q-\sqrt{2m^2g(h-z)}\right)+\delta\left(q+\sqrt{2m^2g(h-z)}\right)\right]\ .$$

Normalizing,

$$\int_0^h \mathrm{d}z \int_{-\infty}^\infty \mathrm{d}q\, P_1(q,z|h) = 2B\int_0^h \frac{\mathrm{d}z}{\sqrt{h-z}} = 2B\int_0^h \frac{\mathrm{d}s}{\sqrt{s}} ,$$

where $s \equiv h - z$. So

$$1 = 2B(2\sqrt{h}) \quad \Longrightarrow \quad B = \frac{1}{4\sqrt{h}} \,,$$

and

$$\boxed{P_1(q,z|h) = \frac{1}{4\sqrt{h(h-z)}}\left[\delta\left(q - \sqrt{2m^2g(h-z)}\right) + \delta\left(q + \sqrt{2m^2g(h-z)}\right)\right] \,.}$$

(c) The answer to (b) gives the probability density for a single particle with specified maximum height h, so now we need only sum over all the particles with a distribution $p(h)$. So

$$\begin{aligned} P(q,z) &= \int_0^\infty \mathrm{d}h\, p(h)\, P_1(q,z|h) \\ &= \int_0^\infty \mathrm{d}h\, p(h)\, \frac{1}{4\sqrt{h(h-z)}}\left[\delta\left(q - \sqrt{2m^2g(h-z)}\right) + \delta\left(q + \sqrt{2m^2g(h-z)}\right)\right] \,. \end{aligned}$$

For $q > 0$ the first δ-function on the RHS contributes, with the constraint

$$q^2 = 2m^2g(h-z) \quad \Longrightarrow \quad h = z + \frac{q^2}{2m^2g} \,.$$

The integration over the δ-function produces the Jacobian factor

$$\int \mathrm{d}x\, \delta(f(x)) = J = \frac{1}{\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_0}} \,,$$

where $f(x_0) = 0$. In this case

$$f(h) = q - \sqrt{2m^2g(h-z)} \quad \Longrightarrow \quad \left|\frac{\mathrm{d}f}{\mathrm{d}h}\right| = \frac{1}{2}\sqrt{\frac{2m^2g}{h-z}} \quad \Longrightarrow \quad J = 2\sqrt{\frac{h-z}{2m^2g}} \,.$$

For $q < 0$, the second δ-function on the RHS contributes, with exactly the same value. So,

$$\boxed{P(q,z) = \frac{1}{2\sqrt{2hm^2g}}\, p(h)\Bigg|_{h = z + \frac{q^2}{2m^2g}} \,.}$$

(d) Starting with the answer from (a),

$$P(q,z) = \frac{N}{\sqrt{2\pi mkT}}\,\frac{mg}{kT}\,e^{-\left[\frac{q^2}{2m}+mgz\right]\Big/kT} ,$$

we use

$$h = z + \frac{q^2}{2m^2 g}$$

to express $p(h)$ as an integral:

$$p(h) = \int_0^h \mathrm{d}z \int_{-\infty}^{\infty} \mathrm{d}q\, P(q,z)\,\delta\left(h - z - \frac{q^2}{2m^2 g}\right) ,$$

so

$$p(h) = \int_{-\infty}^{\infty} \mathrm{d}q\, \frac{N}{\sqrt{2\pi mkT}}\,\frac{mg}{kT} \int_0^h \mathrm{d}z\, e^{-\left[\frac{q^2}{2m}+mgz\right]\Big/kT}\,\delta\left(h - z - \frac{q^2}{2m^2 g}\right) .$$

But

$$\int_0^h \mathrm{d}z\, e^{-\left[\frac{q^2}{2m}+mgz\right]\Big/kT}\,\delta\left(h - z - \frac{q^2}{2m^2 g}\right) = e^{-mgh/kT}\,\theta\left(h - \frac{q^2}{2m^2 g}\right) ,$$

where

$$\theta(x) \equiv \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} .$$

So,

$$\begin{aligned} p(h) &= \frac{N}{\sqrt{2\pi mkT}}\,\frac{mg}{kT}\,e^{-mgh/kT} \int_{-\infty}^{\infty} \mathrm{d}q\,\theta\left(h - \frac{q^2}{2m^2 g}\right) \\ &= \frac{N}{\sqrt{2\pi mkT}}\,\frac{mg}{kT}\,e^{-mgh/kT}\left(2\sqrt{2hm^2 g}\right) \\ &= \boxed{\frac{2N}{\sqrt{\pi}}\left(\frac{mg}{kT}\right)^{3/2}\sqrt{h}\,e^{-mgh/kT} .} \end{aligned}$$

Statistical Mechanics Problem 2
Solution

(a) If the cube has sides of length L, then the spatial wave functions $e^{i\vec{p}\cdot\vec{r}/\hbar}$ must be periodic functions with this period in each direction. Thus,

$$p_x L/\hbar = 2\pi n_x ,$$

where n_x is an integer, with analogous relations for p_y and p_z. Therefore

$$\vec{p} = \frac{2\pi\hbar}{L}\vec{n} ,$$

where n is a vector of integers.

The sum over all states can be represented as a sum over the occupation numbers $\{N_{\vec{p}}\}$ for each of the allowed momenta $\vec{p}$. Since the allowed momenta are countable, we can imagine numbering them as $\vec{p}_j$, where $j = 1, 2, \ldots$. Then

$$Z = \sum_{N_1}\sum_{N_2}\ldots e^{-(E-\mu N)/kT} .$$

Since $N = \sum N_j$ and $E = \sum N_j E_j$, the expression can be written as an infinite product:

$$Z = \sum_{N_1} e^{-(E_1-\mu)N_1/kT} \sum_{N_2} e^{-(E_2-\mu)N_2/kT} \ldots .$$

Then

$$\ln Z = \sum_j \ln\left\{\sum_{N_j} e^{-(E_j-\mu)N_j/kT}\right\} .$$

For bosons, the sum over the occupation numbers N_j extends from 0 to ∞, giving a geometric series:

$$\ln Z_B = \sum_j \ln\left\{\frac{1}{1-e^{-(E_j-\mu)/kT}}\right\} .$$

For fermions, the sum extends only over 0 and 1, so

$$\ln Z_F = \sum_j \ln\left\{1 + e^{-(E_j-\mu)/kT}\right\} .$$

To convert the sum over j to an integral, note that j is really a shorthand for the integer-valued vector $\vec{n}$, so

$$\sum_j = \sum_{n_x}\sum_{n_y}\sum_{n_z} \longrightarrow \int d^3n = \left(\frac{L}{2\pi\hbar}\right)^3 \int d^3p .$$

Comparing with the definition of $f(\vec{p}, m, \mu, T)$, one sees that

$$f(\vec{p}, m, \mu, T) = \begin{cases} -\frac{1}{(2\pi\hbar)^3} \ln\left(1 - e^{-(E_{\vec{p}}-\mu)/kT}\right) & \text{for bosons} \\ \frac{1}{(2\pi\hbar)^3} \ln\left(1 + e^{-(E_{\vec{p}}-\mu)/kT}\right) & \text{for fermions} \end{cases} ,$$

where for relativistic particles

$$E_{\vec{p}} = \sqrt{|\vec{p}|^2 c^2 + m^2 c^4} .$$

For a dilute gas, $e^{-(E_{\vec{p}}-\mu)/kT} \ll 1$, so both expressions reduce to

$$f(\vec{p}, m, \mu, T) = \frac{1}{(2\pi\hbar)^3} e^{-(E_{\vec{p}}-\mu)/kT} .$$

Taking also the nonrelativistic limit for the energy,

$$f_{\text{class}}(\vec{p}, m, \mu, T) = \frac{1}{(2\pi\hbar)^3} e^{-\left(mc^2 + \frac{|\vec{p}|^2}{2m} - \mu\right) / kT} .$$

(b) For this case the integration can be carried out using the formula at the start of the question, so

$$\ln Z_{\text{class}} = \frac{V}{(2\pi\hbar)^3} e^{-(mc^2-\mu)/kT} (2\pi mkT)^{3/2} .$$

From the definition of the partition function, one sees that

$$\frac{1}{Z}\frac{\partial Z}{\partial \mu} = \frac{\langle N \rangle}{kT} ,$$

so

$$n = \frac{\langle N \rangle}{V} = \frac{(2\pi mkT)^{3/2}}{(2\pi\hbar)^3} e^{-(mc^2-\mu)/kT} .$$

(c) The number density of each of the three species can be expressed in terms of its chemical potential by using the answer from part (b):

$$n_n = 2\frac{(2\pi m_n kT)^{3/2}}{(2\pi\hbar)^3} e^{(\mu_n - m_n c^2)/kT}$$

$$n_p = 2\frac{(2\pi m_p kT)^{3/2}}{(2\pi\hbar)^3} e^{(\mu_p - m_p c^2)/kT}$$

$$n_\alpha = \frac{(2\pi m_\alpha kT)^{3/2}}{(2\pi\hbar)^3} e^{(\mu_\alpha - m_\alpha c^2)/kT} .$$

I have included a factor of 2 in the expressions for n_n and n_p, because each particle is spin-$\frac{1}{2}$. Each spin state contributes with the same density as a spinless particle. The ratio is then given by

$$\frac{n_n^2 n_p^2}{n_\alpha} = \frac{m_n^3 m_p^3}{\sqrt{2}\hbar^9 m_\alpha^{3/2}} \left(\frac{kT}{\pi}\right)^{9/2} \exp\left\{-\left[(2m_n + 2m_p - m_\alpha)c^2 + \right.\right.$$
$$\left.\left. + (\mu_\alpha - 2\mu_n - 2\mu_p)\right]/kT\right\} .$$

The binding energy B is defined as the energy that is released when an alpha particle is formed, so

$$2m_n c^2 + 2m_p c^2 = m_\alpha c^2 + B .$$

Since the sum of the chemical potentials on the left-hand side of the reaction equation must equal the sum of the chemical potentials on the right-hand side, we have

$$2\mu_n + 2\mu_p = \mu_\alpha ,$$

where I used the fact that the chemical potential of the photon is necessarily zero, since the photon carries no conserved quantities. Using the two relations above, the expression for the ratio simplifies to

$$\boxed{\frac{n_n^2 n_p^2}{n_\alpha} = \frac{m_n^3 m_p^3}{\sqrt{2}\hbar^9 m_\alpha^{3/2}} \left(\frac{kT}{\pi}\right)^{9/2} e^{-B/kT} .}$$

It is a good approximation to set $m_n = m_p$ and $m_\alpha = 4m_p$ in the prefactor, which gives the simpler expression

$$\boxed{\frac{n_n^2 n_p^2}{n_\alpha} = \frac{1}{8\sqrt{2}\hbar^9} \left(\frac{m_p kT}{\pi}\right)^{9/2} e^{-B/kT} .}$$

Either of the boxed answers is completely acceptable.

Quantum Mechanics Problem 1
Solution

(a) The A and B spins can be described by states of definite total AB spin as

$$|S_{AB}{=}1, M{=}1\rangle = |\uparrow\uparrow\rangle$$

$$|S_{AB}{=}1, M{=}0\rangle = \frac{1}{\sqrt{2}}\Big[\,|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\,\Big]$$

$$|S_{AB}{=}1, M{=}{-1}\rangle = |\downarrow\downarrow\rangle \ ,$$

where the ket vectors on the right are in the basis $|m_A m_B\rangle$, with $m = \pm\frac{1}{2}$ denoted by $\uparrow$ and $\downarrow$. To combine this spin-1 with the spin-$\frac{1}{2}$ of particle C, we first construct the $S_{\rm tot} = 3/2$ states in terms of the $|S_{AB} M s m\rangle$ basis, using the J_- formula given at the start of the question:

$$\left|S_{\rm tot}{=}\tfrac{3}{2}, M{=}\tfrac{3}{2}\right\rangle = \left|1\,1\,\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle$$

$$\sqrt{\tfrac{3}{2}\cdot\tfrac{5}{2} - \tfrac{3}{2}\cdot\tfrac{1}{2}}\ \left|S_{\rm tot}{=}\tfrac{3}{2}, M{=}\tfrac{1}{2}\right\rangle = \sqrt{1\cdot 2 - 1\cdot 0}\ \left|1\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{2}\cdot\tfrac{3}{2} - \tfrac{1}{2}\cdot\left(-\tfrac{1}{2}\right)}\ \left|1\,1\,\tfrac{1}{2}\,{-\tfrac{1}{2}}\right\rangle$$

$$\sqrt{3}\ \left|S_{\rm tot}{=}\tfrac{3}{2}, M{=}\tfrac{1}{2}\right\rangle = \sqrt{2}\ \left|1\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle + \left|1\,1\,\tfrac{1}{2}\,{-\tfrac{1}{2}}\right\rangle \ ,$$

so

$$\left|S_{\rm tot}{=}\tfrac{3}{2}, M{=}\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\ \left|1\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\ \left|1\,1\,\tfrac{1}{2}\,{-\tfrac{1}{2}}\right\rangle \ .$$

By orthogonality,

$$\left|S_{\rm tot}{=}\tfrac{1}{2}, M{=}\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\ \left|1\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\ \left|1\,1\,\tfrac{1}{2}\,{-\tfrac{1}{2}}\right\rangle \ .$$

The quantum state $|\Psi\rangle$ is the $\left|S_{\rm tot}{=}\frac{1}{2}, M{=}\frac{1}{2}\right\rangle$ state, but we need to replace the S_{AB} spin-1 states with their expansion in terms of 2 spin-$\frac{1}{2}$ states, as given above. Thus, in terms of the $|m_A m_B m_C\rangle$ basis,

$$|\Psi\rangle = \sqrt{\tfrac{1}{6}}\ \Big\{\,|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle\,\Big\} - \sqrt{\tfrac{2}{3}}\ |\uparrow\uparrow\downarrow\rangle \ .$$

Summing the probabilities (not the amplitudes!) over the unmeasured degrees of freedom m_B and m_C,

$$P(m_A{=}\uparrow) = \sum_{m_B m_C} |\langle\uparrow m_B m_C|\Psi\rangle|^2 = \frac{1}{6} + \frac{2}{3} = \boxed{\frac{5}{6}\ .}$$

(b) The state $|\Psi\rangle$ is an eigenstate of the z-component of total angular momentum J_z. Therefore it changes by only a phase when rotated about the z-axis, so the expectation value of any operator must be invariant under such rotations. It follows that $\langle\Psi\,|s_{B,x}|\,\Psi\rangle = 0$, so

$$\boxed{P\big(s_{B,x}{=}\uparrow\big) = \frac{1}{2}\ .}$$

(c) The operator corresponding to this measurement is $s_{C,z}$, rotated counterclockwise by θ about the y-axis. Equivalently, we can measure $s_{C,z}$ in a state which is obtained by rotating $|\Psi\rangle$ by an angle $-\theta$ about the y-axis. Since $|\Psi\rangle$ is a spin-$\frac{1}{2}$ state, its rotation can be described by the matrix $R_y(\theta)$ given at the start of the problem. However, we will need to know how to write the $M_{\rm tot}{=}-\frac{1}{2}$ state, which we can find by applying the lowering operator:

$$|\Psi\rangle = \left|S_{\rm tot}{=}\tfrac{1}{2}, M{=}\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\,\left|1\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\,\left|1\,1\,\tfrac{1}{2}\,{-\tfrac{1}{2}}\right\rangle ,$$

so

$$\sqrt{\tfrac{1}{2}\cdot\tfrac{3}{2} - \tfrac{1}{2}\cdot\left(-\tfrac{1}{2}\right)}\,\left|S_{\rm tot}{=}\tfrac{1}{2}, M{=}-\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\left[\sqrt{1\cdot 2 - 0\cdot(-1)}\,\left|1\,{-1}\,\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle \right.$$
$$\left. + \sqrt{\tfrac{1}{2}\cdot\tfrac{3}{2} - \tfrac{1}{2}\cdot\left(-\tfrac{1}{2}\right)}\,\left|1\,0\,\tfrac{1}{2}\,{-\tfrac{1}{2}}\right\rangle\right] - \sqrt{\tfrac{2}{3}}\left[\sqrt{1\cdot 2 - 1\cdot 0}\,\left|1\,0\,\tfrac{1}{2}\,{-\tfrac{1}{2}}\right\rangle\right] ,$$

or

$$\left|S_{\rm tot}{=}\tfrac{1}{2}, M{=}-\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\,\left|1\,{-1}\,\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{1}{3}}\,\left|1\,0\,\tfrac{1}{2}\,{-\tfrac{1}{2}}\right\rangle .$$

Then

$$\begin{aligned}
|\Psi'\rangle &= R_y(-\theta)\,|\Psi\rangle \\
&= \sum_{M,M'} \left|S_{\rm tot}{=}\tfrac{1}{2}, M\right\rangle \left\langle S_{\rm tot}{=}\tfrac{1}{2}, M\right| R_y(-\theta) \left| S_{\rm tot}{=}\tfrac{1}{2}, M'\right\rangle \left\langle S_{\rm tot}{=}\tfrac{1}{2}, M' | \Psi\right\rangle \\
&= \left[\left|S_{\rm tot}{=}\tfrac{1}{2}, M{=}\tfrac{1}{2}\right\rangle, \left|S_{\rm tot}{=}\tfrac{1}{2}, M{=}-\tfrac{1}{2}\right\rangle\right]\left[\cos\tfrac{1}{2}\theta + i\sigma_y \sin\tfrac{1}{2}\theta\right]\begin{bmatrix}1\\0\end{bmatrix} \\
&= \left[\left|S_{\rm tot}{=}\tfrac{1}{2}, M{=}\tfrac{1}{2}\right\rangle, \left|S_{\rm tot}{=}\tfrac{1}{2}, M{=}-\tfrac{1}{2}\right\rangle\right]\begin{bmatrix}\cos\tfrac{1}{2}\theta\\ -\sin\tfrac{1}{2}\theta\end{bmatrix} \\
&= \left|S_{\rm tot}{=}\tfrac{1}{2}, M{=}\tfrac{1}{2}\right\rangle \cos\tfrac{1}{2}\theta \;-\; \left|S_{\rm tot}{=}\tfrac{1}{2}, M{=}-\tfrac{1}{2}\right\rangle \sin\tfrac{1}{2}\theta \\
&= \left[\sqrt{\tfrac{1}{3}}\,\left|1\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{3}}\,\left|1\,1\,\tfrac{1}{2}\,{-\tfrac{1}{2}}\right\rangle\right]\cos\tfrac{1}{2}\theta \\
&\quad - \left[\sqrt{\tfrac{2}{3}}\,\left|1\,{-1}\,\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{1}{3}}\,\left|1\,0\,\tfrac{1}{2}\,{-\tfrac{1}{2}}\right\rangle\right]\sin\tfrac{1}{2}\theta .
\end{aligned}$$

Since we are interested in spin C, there is no need to expand the (S_{AB}, M) labels used here to describe the first two spins. The probability that spin C is up is given by

$$P(m_C = \uparrow) = \textstyle\sum_{S,M} \left|\left\langle S\,M\,\tfrac{1}{2}\,\tfrac{1}{2}|\Psi'\right\rangle\right|^2 = \tfrac{1}{3}\cos^2\left(\tfrac{1}{2}\theta\right) + \tfrac{2}{3}\sin^2\left(\tfrac{1}{2}\theta\right)$$

$$\boxed{= \tfrac{1}{3}\left(1 + \sin^2\tfrac{1}{2}\theta\right) .}$$

(d) The statistical properties of mixed states are completely described by the density matrix. If the system has probability p_i of being described by the state vector $|\psi_i\rangle$, then the density matrix is given by

$$\rho = \sum_i |\psi_i\rangle \; p_i \; \langle\psi_i| \; .$$

For state (i),

$$\rho = \begin{array}{c} \\ \langle m{=}\tfrac{1}{2}| \\ \langle m{=}-\tfrac{1}{2}| \end{array} \begin{array}{c} \quad |m{=}\tfrac{1}{2}\rangle \quad |m{=}-\tfrac{1}{2}\rangle \\ \begin{pmatrix} \tfrac{1}{2} & 0 \\ 0 & \tfrac{1}{2} \end{pmatrix} \end{array} .$$

Since ρ is proportional to the identity matrix, this state is completely unpolarized.

For state (ii),

$$\begin{aligned} \rho &= \frac{1}{4\pi}\int_0^\pi \sin\theta\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \; |\Psi(\theta,\phi)\rangle\,\langle\Psi(\theta,\phi)| \\ &= \frac{1}{4\pi}\int_0^\pi \sin\theta\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \begin{pmatrix} \cos^2\tfrac{1}{2}\theta & \sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta e^{i\phi} \\ \sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta e^{-i\phi} & \sin^2\tfrac{1}{2}\theta \end{pmatrix} \\ &= \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \end{aligned}$$

In carrying out the integration over θ above, note that

$$\int_0^\pi \sin\theta\,\mathrm{d}\theta\,\cos^2\tfrac{1}{2}\theta = \int_0^\pi \sin\theta\,\mathrm{d}\theta \left(\frac{1+\cos\theta}{2}\right) .$$

The $\cos\theta$ term integrates to zero over the two quadrants, so

$$\int_0^\pi \sin\theta\,\mathrm{d}\theta\,\cos^2\tfrac{1}{2}\theta = \frac{1}{2}\int_0^\pi \sin\theta\,\mathrm{d}\theta = 1 .$$

Similarly,

$$\int_0^\pi \sin\theta\,\mathrm{d}\theta\,\sin^2\tfrac{1}{2}\theta = \int_0^\pi \sin\theta\,\mathrm{d}\theta \left(\frac{1-\cos\theta}{2}\right) = 1 .$$

Thus, the density matrix for this state is also proportional to the identity matrix, and so **the two states are indistinguishable.**

Quantum Mechanics Problem 2
Solution

(a) Remembering that $[x, p] = i\hbar$, the Hamiltonian can be factorized as

$$H_0 = \left(\sqrt{\frac{k}{2}}x - \frac{i}{\sqrt{2m}}p\right)\left(\sqrt{\frac{k}{2}}x + \frac{i}{\sqrt{2m}}p\right) + \tfrac{1}{2}\hbar\omega \ ,$$

where

$$\omega = \sqrt{\frac{k}{m}} \ .$$

If we set

$$\boxed{\begin{aligned} a^\dagger &= \frac{1}{\sqrt{\hbar\omega}}\left(\sqrt{\frac{k}{2}}x - \frac{i}{\sqrt{2m}}p\right) \\ a &= \frac{1}{\sqrt{\hbar\omega}}\left(\sqrt{\frac{k}{2}}x + \frac{i}{\sqrt{2m}}p\right) \ , \end{aligned}}$$

then H_0 can be written

$$H_0 = \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right) \ ,$$

as desired. Furthermore

$$\begin{aligned} [a, a^\dagger] &= \frac{1}{\hbar\omega}\left[\left(\sqrt{\frac{k}{2}}x + \frac{i}{\sqrt{2m}}p\right), \left(\sqrt{\frac{k}{2}}x - \frac{i}{\sqrt{2m}}p\right)\right] \\ &= \frac{1}{\hbar\omega} \times 2 \times \sqrt{\frac{k}{2}}\frac{1}{\sqrt{2m}}\hbar = 1 \ , \end{aligned}$$

as desired.

(b) Adding the equations above for a and $a^\dagger$,

$$a + a^\dagger = 2\frac{1}{\sqrt{\hbar\omega}}\sqrt{\frac{k}{2}}\,x \ ,$$

so

$$x = \sqrt{\frac{\hbar\omega}{2k}}\left(a + a^\dagger\right) \ ,$$

and then

$$\begin{aligned} H_1(t) &= \epsilon\frac{\hbar\omega}{2k}e^{-\lambda t}\left(a + a^\dagger\right)^2 \\ &= \epsilon\frac{\hbar\omega}{2k}e^{-\lambda t}\left(a^{\dagger 2} + a^\dagger a + a a^\dagger + a^2\right) \\ &= \boxed{\epsilon\frac{\hbar\omega}{2k}e^{-\lambda t}\left(a^{\dagger 2} + 2a^\dagger a + a^2 + 1\right) \ .} \end{aligned}$$

(c) Time-dependent perturbation theory is most easily carried out in the interaction picture, in which the 0-order time dependence is incorporated into the operators instead of the states. Starting with $|\Psi(t)\rangle$ in the Schrödinger picture, the interaction picture is given by

$$|\Psi_I(t)\rangle \equiv e^{iH_0 t/\hbar}\,|\Psi(t)\rangle \ .$$

Then

$$\begin{aligned} i\hbar\frac{\partial\,|\Psi_I(t)\rangle}{\partial t} &= -H_0 e^{iH_0 t/\hbar}\,|\Psi(t)\rangle + i\hbar e^{iH_0 t/\hbar}\,\frac{\partial\,|\Psi(t)\rangle}{\partial t} \\ &= e^{iH_0 t/\hbar}\left(-H_0 + H_0 + H_1(t)\right)|\Psi(t)\rangle \\ &= e^{iH_0 t/\hbar} H_1(t) e^{-iH_0 t/\hbar}\,|\Psi_I(t)\rangle = \tilde{H}_1(t)\,|\Psi_I(t)\rangle \ , \end{aligned}$$

where

$$\tilde{H}_1(t) = e^{iH_0 t/\hbar} H_1(t) e^{-iH_0 t/\hbar} \ .$$

The perturbative solution to this equation gives

$$|\Psi_I(t_f)\rangle = T\left\{\exp\left(-\frac{i}{\hbar}\int_{t_i}^{t_f}\tilde{H}_1(t)\,\mathrm{d}t\right)\right\}|\Psi_I(t_i)\rangle \ .$$

Here T denotes the time-ordered product, which means that when the exponential is expanded as a power series, the factors of $\tilde{H}_1(t_n)\ldots\tilde{H}_1(t_1)$ that occur in the nth term should always be ordered so that $t_n \geq t_{n-1} \geq \ldots \geq t_1$. To express $\tilde{H}_1(t)$, use the raising and lowering properties of $a^\dagger$ and a to show that

$$\begin{aligned} e^{iH_0 t/\hbar} a e^{-iH_0 t/\hbar} &= a e^{-i\omega t} \\ e^{iH_0 t/\hbar} a^\dagger e^{-iH_0 t/\hbar} &= a^\dagger e^{i\omega t} \ , \end{aligned}$$

so

$$\tilde{H}_1(t) = \epsilon\frac{\hbar\omega}{2k}e^{-\lambda t}\left(a^{\dagger 2}e^{2i\omega t} + 2a^\dagger a + a^2 e^{-2i\omega t} + 1\right) \ .$$

To lowest order in ϵ, at arbitrarily late times,

$$\langle n{=}2|\Psi_I(\infty)\rangle = -\frac{i}{\hbar}A\int_0^\infty e^{-\lambda t}\,e^{2i\omega t}\,\langle n{=}2\,|a^{\dagger 2}|\,n{=}0\rangle\ \mathrm{d}t \ ,$$

where

$$\boxed{A \equiv \epsilon\frac{\hbar\omega}{2k} \ .}$$

Use

$$a^{\dagger n}\,|0\rangle = \sqrt{n!}\,|n\rangle \ ,$$

so

$$\langle n{=}2|\Psi_I(\infty)\rangle = -\frac{i}{\hbar}A\sqrt{2!}\frac{1}{\lambda - 2i\omega} \; .$$

The probability is then

$$\boxed{P(n{=}2) = \frac{2A^2}{\hbar^2(\lambda^2 + 4\omega^2)} \; ,}$$

where A is given by the boxed equation above.

(d) Since $\tilde{H}_I(t)$ changes the quantum number n only by 0 or ± 2, only even values of n are ever achieved. So

$$\boxed{P(n) = 0 \text{ if } n \text{ is odd.}}$$

The state $|2m\rangle$ is reached in lowest nonvanishing order by the mth term of the expansion, where only the term in $\tilde{H}_I(t)$ proportional to $a^{\dagger 2}$ contributes. Since this term commutes with itself, the ordering of factors is irrelevant, so to this order in perturbation theory we can ignore the time-ordering of the product. Thus,

$$\begin{aligned}\langle 2m|\Psi_I(\infty)\rangle &= \frac{(-1)^m}{m!}\left\{\frac{i}{\hbar}A\int_0^\infty e^{-\lambda t}\, e^{2i\omega t}\, dt\right\}^m \left\langle 2m \left| a^{\dagger 2m} \right| n{=}0\right\rangle \\ &= \frac{(-1)^m}{m!}\left(\frac{A}{\hbar(\lambda - 2i\omega)}\right)^m \sqrt{(2m)!} \; .\end{aligned}$$

Finally,

$$\boxed{P(2m) = \frac{(2m)!}{m!^2}\left(\frac{A^2}{\hbar^2(\lambda^2 + 4\omega^2)}\right)^m \; .}$$