General Exam Fall 2001 Solutions

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Mechanics Problem 1

(a) The Lagrangian $\mathcal{L}$ for this system is written in terms of the position vectors $\mathbf{x}_i$ of the masses and their time derivatives $\dot{\mathbf{x}}_i$.

$$\mathcal{L} = T - V = \frac{1}{2}\left(M_1\dot{\mathbf{x}}_1^2 + M_2\dot{\mathbf{x}}_2^2 + M_3\dot{\mathbf{x}}_3^2\right) + G\left(\frac{M_1M_2}{|\mathbf{x}_1 - \mathbf{x}_2|} + \frac{M_2M_3}{|\mathbf{x}_2 - \mathbf{x}_3|} + \frac{M_1M_3}{|\mathbf{x}_1 - \mathbf{x}_3|}\right) \tag{1}$$

The center-of-mass of the $M_1 + M_2$ binary has position vector $\mathbf{R}$:

$$(M_1 + M_2)\mathbf{R} = M_1\mathbf{x}_1 + M_2\mathbf{x}_2 \tag{2}$$

The separation vector between the M_1 and M_2 is $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$. We can solve for $\mathbf{x}_1$ and $\mathbf{x}_2$ in terms of $\mathbf{R}$, $\mathbf{r}$, and the masses:

$$\begin{cases} \mathbf{x}_1 = \mathbf{R} + \left(\dfrac{M_2}{M_1+M_2}\right)\mathbf{r} \\ \mathbf{x}_2 = \mathbf{R} - \left(\dfrac{M_1}{M_1+M_2}\right)\mathbf{r} \\ \mathbf{x}_3 = \mathbf{R} + \mathbf{s} \end{cases} \tag{3}$$

(The last line is just the definition given in the problem statement for the position of M_3.) Let's consider the kinetic T and potential V parts of the Lagrangian separately.

$$\begin{aligned} T &= \frac{1}{2}\left(M_1\dot{\mathbf{x}}_1^2 + M_2\dot{\mathbf{x}}_2^2 + M_3\dot{\mathbf{x}}_3^2\right) && (4)\\ &= \frac{1}{2}\left(M_1+M_2\right)\dot{\mathbf{R}}^2 + \frac{1}{2}\left(\frac{M_1M_2}{M_1+M_2}\right)\dot{\mathbf{r}}^2 + \frac{1}{2}M_3\left(\dot{\mathbf{R}}^2 + 2\dot{\mathbf{R}}\cdot\dot{\mathbf{s}} + \dot{\mathbf{s}}^2\right) && (5)\\ &= \frac{1}{2}(M_1+M_2)\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 + \frac{1}{2}M_3\left(\dot{\mathbf{R}}^2 + 2\dot{\mathbf{R}}\cdot\dot{\mathbf{s}} + \dot{\mathbf{s}}^2\right) && (6) \end{aligned}$$

Now, onto the potentials:

$$\begin{aligned} \frac{1}{|\mathbf{x}_1 - \mathbf{x}_3|} &= \frac{1}{\left|\mathbf{s} - \left(\frac{M_2}{M_1+M_2}\right)\mathbf{r}\right|} && (7)\\ &= \frac{1}{s}\left[1 + \left(\frac{M_2}{M_1+M_2}\right)\frac{\mathbf{r}\cdot\mathbf{s}}{s^2} + \mathcal{O}\left(\frac{r^2}{s^2}\right)\right] && (8)\\ \frac{1}{|\mathbf{x}_2 - \mathbf{x}_3|} &= \frac{1}{\left|\mathbf{s} + \left(\frac{M_1}{M_1+M_2}\right)\mathbf{r}\right|} && (9)\\ &= \frac{1}{s}\left[1 - \left(\frac{M_1}{M_1+M_2}\right)\frac{\mathbf{r}\cdot\mathbf{s}}{s^2} + \mathcal{O}\left(\frac{r^2}{s^2}\right)\right] && (10) \end{aligned}$$

Thus we have

$$\mathcal{L} = \frac{1}{2}(M_1+M_2)\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 + \frac{1}{2}M_3\left(\dot{\mathbf{R}}^2 + 2\dot{\mathbf{R}}\cdot\dot{\mathbf{s}} + \dot{\mathbf{s}}^2\right) + \frac{GM_1M_2}{r} + \frac{G(M_1+M_2)M_3}{s}\left[1 + \mathcal{O}\left(\frac{r^2}{s^2}\right)\right] \tag{11}$$

To convert to the Hamiltonian $\mathcal{H}$, we first need to calculate the corresponding momenta:

$$\begin{cases} \mathbf{p}_R = \dfrac{\partial\mathcal{L}}{\partial\dot{\mathbf{R}}} = (M_1+M_2)\dot{\mathbf{R}} + M_3(\dot{\mathbf{R}} + \dot{\mathbf{s}}) \\ \mathbf{p}_r = \dfrac{\partial\mathcal{L}}{\partial\dot{\mathbf{r}}} = \mu\dot{\mathbf{r}} \\ \mathbf{p}_s = \dfrac{\partial\mathcal{L}}{\partial\dot{\mathbf{s}}} = M_3(\dot{\mathbf{R}} + \dot{\mathbf{s}}) \end{cases} \tag{12}$$

Next, we take the Legendre transformation $h = \sum_i p_i \dot{q}_i - \mathcal{L}$ where p_i is the momentum component we just found and $\dot{q}$ is the time derivative of the corresponding position coordinate. Thus

$$h(\mathbf{R},\mathbf{r},\mathbf{s};\dot{\mathbf{R}},\dot{\mathbf{r}},\dot{\mathbf{s}}) = (M_1+M_2+M_3)\dot{\mathbf{R}}^2 + 2M_3\dot{\mathbf{R}}\cdot\dot{\mathbf{s}} + \mu\dot{\mathbf{r}}^2 + M_3\dot{\mathbf{s}}^2 - \mathcal{L} \tag{13}$$

$$= \frac{1}{2}(M_1+M_2)\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 + \frac{1}{2}M_3\left(\dot{\mathbf{R}}^2 + 2\dot{\mathbf{R}}\cdot\dot{\mathbf{s}} + \dot{\mathbf{s}}^2\right) - \frac{GM_1M_2}{r} - \frac{G(M_1+M_2)M_3}{s}\left[1+\mathcal{O}\left(\frac{r^2}{s^2}\right)\right] \tag{14}$$

(The reason I symbolized it h is because it depends on the position variables and their time derivatives, whereas the actual Hamiltonian $\mathcal{H}$ depends on the position variables and their corresponding momenta.) Now solve for $(\dot{\mathbf{R}},\dot{\mathbf{r}},\dot{\mathbf{s}})$ in terms of $(\mathbf{p}_R,\mathbf{p}_r,\mathbf{p}_s)$:

$$\begin{cases} \dot{\mathbf{R}} = \dfrac{\mathbf{p}_R - \mathbf{p}_s}{M_1+M_2} \\ \dot{\mathbf{r}} = \dfrac{\mathbf{p}_r}{\mu} \\ \dot{\mathbf{s}} = \dfrac{\mathbf{p}_s}{M_3} + \dfrac{\mathbf{p}_s - \mathbf{p}_r}{M_1+M_2} \end{cases} \tag{15}$$

Substituting these into (14), we obtain the final result for $\mathcal{H}$:

$$\boxed{\mathcal{H}(\mathbf{R},\mathbf{r},\mathbf{s};\mathbf{p}_R,\mathbf{p}_r,\mathbf{p}_s) = \frac{1}{2}\frac{(\mathbf{p}_R-\mathbf{p}_s)^2}{(M_1+M_2)^2} + \frac{\mathbf{p}_r^2}{2\mu} + \frac{\mathbf{p}_s^2}{2M_3} - \frac{GM_1M_2}{r} - \frac{G(M_1+M_2)M_3}{s}\left[1+\mathcal{O}\left(\frac{r^2}{s^2}\right)\right]} \tag{16}$$

(b) Our Hamiltonian is separable:

$$\mathcal{H} = \mathcal{H}_r(\mathbf{r},\mathbf{p}_r) + \mathcal{H}_{RS}(\mathbf{R},\mathbf{s};\mathbf{p}_R,\mathbf{p}_s) \tag{17}$$

This means we can solve for the r-motion independent of R and s. We know $\mathcal{H}_r = \mathbf{p}_r^2/2\mu - GM_1M_2/r$ is just the standard 2-body Keplerian Hamiltonian, for which the trajectories are conic sections with eccentricity e:

$$r(\phi) = \frac{a(1-e^2)}{1+e\cos\phi}. \tag{18}$$

For a bound orbit ($E<0$), the turning points occur at $\phi = 0$ and $\phi = \pi$, corresponding to separations $r_\pm = a(1\pm e)$. At these values of ϕ, the velocity vector of the objects is entirely tangential to their orbit; *i.e.*, there is no radial component to their velocity. I should note that $\mathbf{p}_r$ is the *total* momentum vector corresponding to the binary orbit, and is not just the radial component of the orbit. Since the binary's angular momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}_r$ is conserved, we can evaluate it at the turning points, where the momenta of the objects are purely tangential: $\mathbf{p}_r \propto \hat{\boldsymbol{\phi}}$. Thus

$$p_{r,\pm} = \frac{L}{r_\pm} = \frac{L}{a(1\pm e)} \tag{19}$$

We also have the total energy E of the binary, now written in terms of L:

$$E = \mathcal{H}_r = \frac{L^2}{2\mu r^2} - \frac{GM_1M_2}{r} \quad \text{for} \quad r = a(1\pm e) \tag{20}$$

Solve for r in terms of the other quantities:

$$0 = r^2 + \frac{GM_1M_2 r}{E} - \frac{L^2}{2\mu E} \tag{21}$$

$$r = -\frac{GM_1M_2}{2E} \pm \sqrt{\left(\frac{GM_1M_2}{2E}\right)^2 + \frac{L^2}{2\mu E}} \tag{22}$$

Since $r = a(1\pm e)$ at the turning points, we can immediately read off

$$\boxed{a = -\frac{GM_1M_2}{2E}, \quad e = \sqrt{1+\frac{2EL^2}{\mu(GM_1M_2)^2}}} \tag{23}$$

(c) From Hamilton's equations,

$$\frac{\partial\mathcal{H}}{\partial\mathbf{R}} = 0 \Rightarrow \frac{d\mathbf{p}_R}{dt} = 0 \tag{24}$$

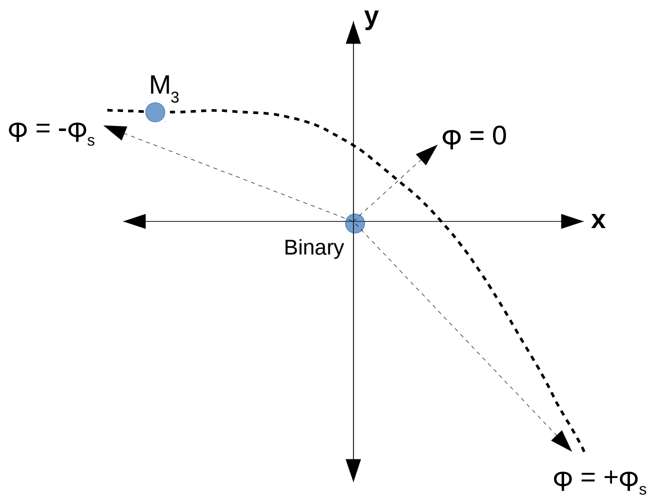

Figure 1: Coordinates for **(1d)**

where

$$\mathbf{p}_R = (M_1+M_2)\dot{\mathbf{R}} + M_3(\dot{\mathbf{R}}+\dot{\mathbf{s}}) = M_1\dot{\mathbf{x}}_1 + M_2\dot{\mathbf{x}}_2 + M_3\dot{\mathbf{x}}_3 \tag{25}$$

Without loss of generality, we can move into the frame in which the CM is at rest; *i.e.*, we can set $\mathbf{p}_R = \mathbf{0}$. Under this simplification, we find

$$\mathcal{H}_{Rs} = \frac{\mathbf{p}_s^2}{2\mu_s} - \frac{G(M_1+M_2)M_3}{s} \quad \text{where} \quad \frac{1}{\mu_s} = \frac{1}{M_1+M_2} + \frac{1}{M_3} \tag{26}$$

$\mathcal{H}_{Rs}$ has exactly the same form as $\mathcal{H}_r$; thus, the $\mathbf{s}$-motion is also Keplerian. Explicitly,

$$\dot{\mathbf{p}}_s = -\frac{\partial\mathcal{H}}{\partial\mathbf{s}} = -\frac{G(M_1+M_2)M_3}{s^3}\mathbf{s} \tag{27}$$

Since $\mathbf{p}_s = \mu_s\dot{\mathbf{s}}$, we have $\dot{\mathbf{p}}_s = \mu_s\ddot{\mathbf{s}}$ and hence

$$\ddot{\mathbf{s}} = -\frac{G(M_1+M_2+M_3)}{s^3}\mathbf{s} \tag{28}$$

We can obtain the orbital parameters a_s and e_s by comparing $\mathcal{H}_{Rs}(\mathbf{p}_R = \mathbf{0}) = E_s$ with $\mathcal{H}_r$, and noting $(M_1+M_2)M_3 = \mu_s(M_1+M_2+M_3)$:

$$\boxed{a_s = -\frac{G(M_1+M_2)M_3}{2E_s}, \quad e_s = \sqrt{1+\frac{2E_sL_s^2}{\mu_s[G(M_1+M_2)M_3]^2}}} \tag{29}$$

(d) Initially, the incoming M_3 has velocity vector $\dot{\mathbf{s}}(t\to-\infty) = v_\infty\hat{\mathbf{x}}$, and we have the freedom to align our coordinate system such that $\hat{\mathbf{x}}$ points along $\dot{\mathbf{s}}$ (see Figure 1). The momentum is $\mathbf{p}_s = \mu_s v_\infty\hat{\mathbf{x}}$. Since

$$s(\phi) = \frac{a_s(1-e_s^2)}{1+e_s\cos\phi_s}, \tag{30}$$

the $s\to\infty$ condition implies $e_s\cos\phi_s = -1$. M_3 initially comes in at the angle $\phi = -\phi_s$ and leaves at $\phi = +\phi_s$; hence, after scattering, M_3 has velocity

$$\dot{\mathbf{s}}(t\to\infty) = v_\infty\left[\hat{\mathbf{x}}\cos(2\phi_s-\pi) - \hat{\mathbf{y}}\sin(2\phi_s-\pi)\right] \tag{31}$$

$$= v_\infty\left[-\hat{\mathbf{x}}\cos(2\phi_s) + \hat{\mathbf{y}}\sin(2\phi_s)\right] \tag{32}$$

Using $\cos\phi_s = -1/e_s$ from above, and some double-angle trig identities, we find

$$\Delta\mathbf{p}_s = \mu_s[\dot{\mathbf{s}}(t\to\infty) - \dot{\mathbf{s}}(t\to-\infty)] \tag{33}$$

$$= \mu_s v_\infty\left[\hat{\mathbf{x}}\left(1-\frac{2}{e_s^2}\right) - \hat{\mathbf{y}}\frac{2}{e_s}\sqrt{1-\frac{1}{e_s^2}}\right] - \mu_s v_\infty\hat{\mathbf{x}} \tag{34}$$

$$= -2\mu_s v_\infty\left[\frac{\hat{\mathbf{x}}}{e_s^2} + \frac{\hat{\mathbf{y}}}{e_s}\sqrt{1-\frac{1}{e_s^2}}\right] \tag{35}$$

This impulse $\Delta\mathbf{p}_s$ is picked up by the binary:

$$(M_1 + M_2)\dot{\mathbf{R}} = -\Delta\mathbf{p}_s \tag{36}$$

In the initial rest frame of the binary, the binary recoils with velocity

$$\dot{\mathbf{R}} = \frac{2\mu_s v_\infty}{M_1 + M_2}\left[\frac{\hat{\mathbf{x}}}{e_s^2} + \frac{\hat{\mathbf{y}}}{e_s}\sqrt{1 - \frac{1}{e_s^2}}\right] \tag{37}$$

and the magnitude of this vector is

$$\boxed{\dot{R} = \frac{2\mu_s v_\infty}{(M_1 + M_2)e_s} = \frac{2M_3 v_\infty}{(M_1 + M_2 + M_3)e_s}} \tag{38}$$

Mechanics Problem 2

(a) Consider the coordinate system shown in Figure 2. The (unit) tangent vector $\hat{\boldsymbol{\tau}}$ to a point $\mathbf{r} = x\hat{\mathbf{x}} + y(x)\hat{\mathbf{y}}$ on the curve is given by

$$\hat{\boldsymbol{\tau}}(x) = \frac{\mathbf{r}'(x)}{|\mathbf{r}'(x)|} = \frac{\hat{\mathbf{x}} + \sinh(x/a)\hat{\mathbf{y}}}{\sqrt{1+\sinh^2(x/a)}} = \frac{\hat{\mathbf{x}}}{\cosh(x/a)} + \tanh(x/a)\hat{\mathbf{y}} \tag{39}$$

We can also calculate this another way: since the arc length s along the curve[1] is given by $ds = dx\,\sqrt{1+(dy/dx)^2}$, we can

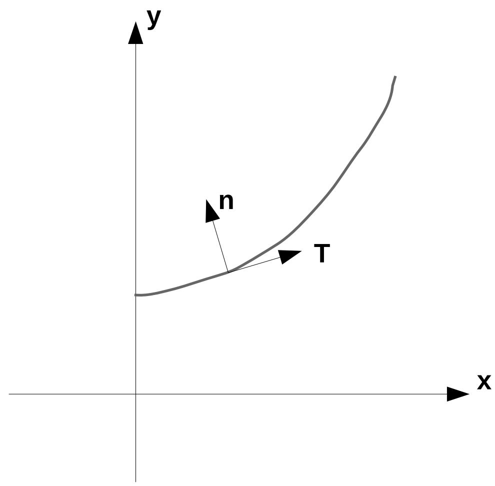

Figure 2: Coordinates for **(2a)**.

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$$\hat{\boldsymbol{\tau}}(x) = \frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\hat{\mathbf{x}} + \frac{dy}{ds}\hat{\mathbf{y}} \tag{40}$$

Using $y(x) = a\cosh(x/a)$, we arrive at

$$\hat{\boldsymbol{\tau}}(x) = \frac{\hat{\mathbf{x}}}{\cosh(x/a)} + \tanh(x/a)\hat{\mathbf{y}}. \tag{41}$$

To construct the (unit) normal vector $\hat{\mathbf{n}}$, we can either use the definition $\hat{\mathbf{n}} = \hat{\boldsymbol{\tau}}'(x)/|\hat{\boldsymbol{\tau}}'(x)|$, or just find the appropriate vector which is orthogonal to $\hat{\boldsymbol{\tau}}$. Either way, we have

$$\hat{\mathbf{n}} = -\tanh(x/a)\hat{\mathbf{x}} + \frac{\hat{\mathbf{y}}}{\cosh(x/a)} \tag{42}$$

The "rolling without slipping" condition means that if the disk rotates through an angle φ, it travels a distance $s = R\varphi$ along the curve. This means

$$s = \int_0^{x_0} dx\,\sqrt{1+(dy/dx)^2} \Rightarrow a\sinh(x_0/a) = R\varphi \tag{43}$$

where (x_0, y_0) are the coordinates of the contact point of the disk and curve:

$$\begin{cases} x_0 = a\sinh^{-1}(R\varphi/a) \\ y_0 = a\cosh(x_0/a) = \sqrt{a^2+R^2\varphi^2} \end{cases} \tag{44}$$

The coordinates (x, y) of the center of the disk are simply given by $(x_0 + Rn_x, y_0 + Rn_y)$:

$$\boxed{\begin{cases} x = a\sinh^{-1}(R\varphi/a) - \dfrac{R^2\varphi}{\sqrt{a^2+R^2\varphi^2}} \\ y = \sqrt{a^2+R^2\varphi^2} + \dfrac{aR}{\sqrt{a^2+R^2\varphi^2}} \end{cases}} \tag{45}$$

[1] We'll need the arc length later anyway.

(b) The Lagrangian $\mathcal{L}$ of the disk is given by

$$\mathcal{L} = \frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}I\dot{\varphi}^2 - Mgy \tag{46}$$

Since x and y are functions of $\varphi(t)$, we have $\dot{x} = x'(\varphi)\dot{\varphi}$ and $\dot{y} = y'(\varphi)\dot{\varphi}$. The moment of inertia of the disk is $I = MR^2/2$. Putting it all together, we arrive at

$$\boxed{\mathcal{L} = \frac{1}{2}MR^2\dot{\varphi}^2\left(D^2 + \frac{1}{2}\right) - Mg\left(\sqrt{a^2 + R^2\varphi^2} + \frac{aR}{\sqrt{a^2 + R^2\varphi^2}}\right), \quad \text{where} \quad D \equiv 1 - \frac{aR}{a^2 + R^2\varphi^2}} \tag{47}$$

(c) For $R^2\varphi^2 \ll a^2$, we have $D \simeq D_0 \equiv 1 - (R/a)$. Thus

$$\mathcal{L} \simeq \frac{1}{2}MR^2\dot{\varphi}^2\left(D_0^2 + \frac{1}{2}\right) - Mg(a + R) - \frac{MgR^2D_0}{2a}\varphi^2 \tag{48}$$

$$= \frac{1}{2}A\dot{\varphi}^2 - \frac{1}{2}B\varphi^2 + \text{const.} \tag{49}$$

which is exactly the Lagrangian of a simple harmonic oscillator. The equation of motion is

$$\ddot{\varphi} = -\frac{B}{A}\varphi \equiv -\omega^2\varphi, \tag{50}$$

and thus the angular frequency of small oscillations is

$$\boxed{\omega = \sqrt{\frac{g}{a}\frac{D_0}{D_0^2 + \frac{1}{2}}}, \quad \text{where} \quad D_0 = 1 - \frac{R}{a}} \tag{51}$$

(d) Newton's second law for this object is

$$M\ddot{\mathbf{x}} = -Mg\hat{\mathbf{y}} + N\hat{\mathbf{n}} + T\hat{\boldsymbol{\tau}} \tag{52}$$

where N and T are the magnitudes of the normal and tangential forces. In component form,

$$\begin{cases} \ddot{x} = \dfrac{d}{dt}\left(\dfrac{dx}{d\varphi}\dot{\phi}\right) = \dfrac{dx}{d\phi}\ddot{\varphi} + \cancel{\dfrac{d^2x}{d\varphi^2}\dot{\varphi}^2} \\ \ddot{y} = \dfrac{d}{dt}\left(\dfrac{dy}{d\varphi}\dot{\phi}\right) = \dfrac{dy}{d\phi}\ddot{\varphi} + \cancel{\dfrac{d^2y}{d\varphi^2}\dot{\varphi}^2} \end{cases} \tag{53}$$

where the $\mathcal{O}(\dot{\varphi}^2)$ terms vanish since $\dot{\varphi} = 0$. Looking back at **(a)**, we see that $n_x = -\tau_y$ and $n_y = \tau_x$. Newton's 2nd law thus reads

$$\begin{cases} M\ddot{x} = -\tau_y N + \tau_x T \\ M\ddot{y} = \tau_x N + \tau_y T - Mg \end{cases} \tag{54}$$

and in matrix form, this is

$$\begin{pmatrix} -\tau_y & \tau_x \\ \tau_x & \tau_y \end{pmatrix}\begin{pmatrix} N \\ T \end{pmatrix} = M\begin{pmatrix} \frac{dx}{d\varphi}\ddot{\varphi} \\ \frac{dy}{d\varphi}\ddot{\varphi} + g \end{pmatrix} \tag{55}$$

Helpfully, the τ-matrix is its own inverse, so

$$\begin{pmatrix} N \\ T \end{pmatrix} = M\begin{pmatrix} -\tau_y & \tau_x \\ \tau_x & \tau_y \end{pmatrix}\begin{pmatrix} \frac{dx}{d\varphi}\ddot{\varphi} \\ \frac{dy}{d\varphi}\ddot{\varphi} + g \end{pmatrix} \tag{56}$$

Using the expressions for τ_x and τ_y in terms of φ,

$$\begin{cases} \tau_x = \dfrac{a}{\sqrt{a^2 + R^2\varphi^2}} \\ \tau_y = \dfrac{R\varphi}{\sqrt{a^2 + R^2\varphi^2}}, \end{cases} \tag{57}$$

we have

$$\begin{cases} \dfrac{N}{M} = -\dfrac{aR^2\varphi D}{a^2+R^2\varphi^2}\ddot{\varphi} + \dfrac{aR^2\dot{\varphi}D}{a^2+R^2\varphi^2}\ddot{\varphi} + \dfrac{ag}{\sqrt{a^2+R^2\varphi^2}} = \dfrac{ga}{\sqrt{a^2+R^2\varphi^2}} \\ \dfrac{T}{m} = \dfrac{a^2RD}{a^2+R^2\varphi^2}\ddot{\varphi} + \dfrac{R^3\varphi^2D}{a^2+R^2\varphi^2}\ddot{\varphi} + \dfrac{gR\varphi}{\sqrt{a^2+R^2\varphi^2}} = RD\ddot{\varphi} + \dfrac{gR\varphi}{\sqrt{a^2+R^2\varphi^2}} \end{cases} \tag{58}$$

For small oscillations, $\ddot{\varphi} = -\omega^2\varphi$ with ω as calculated in **(c)**. We thus find

$$\boxed{\begin{cases} N \simeq Mg\left(1 - \dfrac{R^2\varphi^2}{2a^2}\right) \\ T \simeq \dfrac{MgR\varphi}{a(2D_0^2+1)} \end{cases}} \tag{59}$$

(e) The disk remains stationary if $T < \mu N$. Assuming $R\varphi/a \ll 1$, so we can neglect any terms second-order in φ^2, we have (using $\mu = 1/2$)

$$\frac{MgR\varphi}{a(2D_0^2+1)} < \frac{1}{2}Mg \quad \Rightarrow \quad \frac{R\varphi}{a} < D_0^2 + \frac{1}{2} \tag{60}$$

Since we assumed $R\varphi/a \ll 1$, the disk will not slip, as $D_0^2 + \frac{1}{2} \geq \frac{3}{2} \gg R\varphi/a$.

Electromagnetism Problem 1

(a) Since we are working in Minkowski spacetime in Cartesian coordinates, we can replace $\nabla_\mu = \partial/\partial x^\mu \equiv \partial_\mu$ and write (defining the d'Alembert operator $\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$)

$$\partial^\nu F_{\mu\nu} = \partial^\nu \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) = -4\pi J^\mu \tag{61}$$
$$= \partial_\mu \partial^\nu A_\nu - \Box A^\mu \tag{62}$$

We recover the wave equation $\Box A^\mu = 4\pi J^\mu$ if we impose the (Lorenz) gauge condition $\boxed{\partial^\nu A_\nu = 0}$. The potential A^μ determined is not unique because we can introduce introduce a scalar field $\psi(x)$ such that $A^\mu \to A^\mu + \partial^\mu \psi$, and as long as $\Box\psi = 0$, this will have no effect on the Lorenz gauge condition

$$\partial^\nu \left(A_\nu + \partial_\nu \psi\right) = \partial^\nu A_\nu + \cancel{\Box\psi} = 0 \tag{63}$$

or the field-strength tensor

$$F_{\mu\nu} \to \partial_\mu \left(A_\nu + \partial_\nu \psi\right) - \partial_\nu \left(A_\mu + \partial_\mu \psi\right) = \partial_\mu A_\nu - \partial_\nu A_\mu + \cancel{\left(\partial_\mu \partial_\nu \psi - \partial_\nu \partial_\mu \psi\right)} = F_{\mu\nu} \tag{64}$$

(b) The point charge corresponds to a charge density $\rho = q\delta^3(\mathbf{x} - \mathbf{x}_q(t))$ and current density $\mathbf{J} = q\mathbf{v}_q \delta^3(\mathbf{x} - \mathbf{x}_q(t)$, where $\mathbf{v}_q = d\mathbf{x}_q/dt$. This gives us the 4-current

$$(J^\mu) = (\rho, \mathbf{J}) = q(1, \mathbf{v}_q)\delta^3(\mathbf{x} - \mathbf{x}_q(t)). \tag{65}$$

Unfortunately, the δ-function doesn't help us much in

$$(A^\mu) = \int \frac{q(1, \mathbf{v}_q(t_r))}{|\mathbf{x} - \mathbf{x}'|}\, \delta^3(\mathbf{x} - \mathbf{x}_q(t_r))\, d^3\mathbf{x}' \tag{66}$$

since t_r depends on $\mathbf{x}$, and hence $\mathbf{x}_q(t_r)$ depends on $\mathbf{x}'$. We can work around this by introducing a δ-function in time:

$$(A^\mu) = q \int dt' \int d^3\mathbf{x}' \frac{(1, \mathbf{v}_q(t'))}{|\mathbf{x} - \mathbf{x}'|}\, \delta^3(\mathbf{x}' - \mathbf{x}_q(t'))\delta(t' - t + |\mathbf{x} - \mathbf{x}'|) \tag{67}$$

Now that everything is in terms of primed coordinates, we can do the integral over $d^3\mathbf{x}'$:

$$(A^\mu) = q \int dt' \frac{(1, \mathbf{v}_q(t'))}{|\mathbf{x} - \mathbf{x}_q(t')|}\, \delta(t' - t + |\mathbf{x} - \mathbf{x}_q(t')|) \tag{68}$$

Next, use the δ-function relation

$$\delta(f(t')) = \sum_{t_i \in \{\text{zeros of } f\}} \frac{\delta(t' - t_i)}{|f'(t_i)|} \tag{69}$$

along with

$$\frac{d}{dt'}\left[t' - t + |\mathbf{x} - \mathbf{x}_q(t')|\right] = 1 - \mathbf{v}_q(t') \cdot \frac{\mathbf{x} - \mathbf{x}_q(t')}{|\mathbf{x} - \mathbf{x}_q(t')|} \equiv 1 - \hat{\mathbf{n}} \cdot \mathbf{v}_q(t') \tag{70}$$

Thus

$$\boxed{(A^\mu) = \left[\frac{q(1, \mathbf{v}_q)}{|\mathbf{x} - \mathbf{x}_q|(1 - \mathbf{n} \cdot \mathbf{v}_q)}\right]_{\text{ret}} = \left[\frac{q(1, \mathbf{v}_q)}{R - \mathbf{R} \cdot \mathbf{v}_q}\right]_{\text{ret}}} \tag{71}$$

(c) Assume that the circular motion is taking place in the xy plane. The Lorentz force is already Lorentz-covariant, so we can just use that. (I'm using SI units here.) The 3-momentum of a relativistic particle is $\mathbf{p} = \gamma m \mathbf{v}$, where γ is the Lorentz factor and $\mathbf{v} = d\mathbf{x}/dt$ is the usual 3-velocity. In circular motion, the magnitude of the 3-momentum $p = |\mathbf{p}|$ is constant, so $\mathbf{v}$ and γ are constants as well. Thus

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \gamma m \frac{d\mathbf{v}}{dt} = -eB_z \mathbf{v} \times \hat{\mathbf{z}} \tag{72}$$

This gives two coupled equations

$$\begin{cases} \ddot{x} = -\left(\dfrac{eB_z}{m\gamma}\right)\dot{y} \\ \ddot{y} = +\left(\dfrac{eB_z}{m\gamma}\right)\dot{x} \end{cases} \tag{73}$$

Substituting one into the other, we find that the motion is simple harmonic, with the general equation of motion

$$\ddot{x}_i = -\left(\frac{eB_z}{m\gamma}\right)^2 x_i = -\omega^2 x_i \tag{74}$$

Thus the angular frequency of the orbit is given by

$$\boxed{\omega = \frac{|e|B_z}{m\gamma}} \tag{75}$$

For circular motion with radius a we have (in coordinate time—*i.e.*, the lab frame—not proper time), $v = a\omega$. From $p = \gamma m v$, we find the radius of curvature of the orbit:

$$\boxed{a = \frac{v}{\omega} = \frac{p}{\gamma m}\frac{m\gamma}{|e|B_z} = \frac{p}{|e|B_z}} \tag{76}$$

(d) Coming soon. The original solutions curiously stop after part **(b)**.

Electromagnetism Problem 2

(a) We will need the wave equations for **E** and **B** inside the waveguide, which we model as a perfect conductor. Start with Maxwell's equations:

$$\begin{cases} \nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\dfrac{1}{c}\dfrac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \dfrac{1}{c}\dfrac{\partial \mathbf{E}}{\partial t} \end{cases} \tag{77}$$

Using the general result from vector calculus $\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}$, we find **E** and **B** independently satisfy the free-space wave equation inside the waveguide:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix} \tag{78}$$

As an *ansatz*, consider plane wave solutions

$$\begin{bmatrix}\mathbf{E}\\ \mathbf{B}\end{bmatrix} = \begin{bmatrix}\mathbf{E}_0\\ \mathbf{B}_0\end{bmatrix} \exp\left[i(\mathbf{k}\cdot\mathbf{x} - \omega t)\right] \tag{79}$$

where $\mathbf{E}_0$ and $\mathbf{B}_0$ depend on position. For the sake of argument, let the rectangular cross-section of the waveguide be in the xy plane, and let its longitudinal extent be along the z-axis. From our plane-wave ansatz, this means we can make the replacements

$$\frac{\partial}{\partial t} \to -i\omega \quad \text{and} \quad \frac{\partial}{\partial z} = ik. \tag{80}$$

Using the curl Maxwell equations, we find

$$\begin{cases} \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = \dfrac{i\omega}{c}B_z, & \dfrac{\partial B_y}{\partial x} - \dfrac{\partial B_x}{\partial y} = -\dfrac{i\omega}{c}E_z, \\ \dfrac{\partial E_z}{\partial y} - ikE_y = \dfrac{i\omega}{c}B_x, & \dfrac{\partial B_z}{\partial y} - ikB_y = -\dfrac{i\omega}{c}E_x, \\ ikE_x - \dfrac{\partial E_z}{\partial x} = \dfrac{i\omega}{c}B_y, & ikB_x - \dfrac{\partial B_z}{\partial x} = -\dfrac{i\omega}{c}E_y \end{cases} \tag{81}$$

which we can solve (also using the divergence equations) to find the xy fields in terms of the z fields:

$$\begin{cases} E_x = \dfrac{i}{(\omega/c)^2 - k^2}\left(k\dfrac{\partial E_z}{\partial x} + \dfrac{\omega}{c}\dfrac{\partial B_z}{\partial y}\right) \\ E_y = \dfrac{i}{(\omega/c)^2 - k^2}\left(k\dfrac{\partial E_z}{\partial y} - \dfrac{\omega}{c}\dfrac{\partial B_z}{\partial x}\right) \\ B_x = \dfrac{i}{(\omega/c)^2 - k^2}\left(k\dfrac{\partial B_z}{\partial x} - \dfrac{\omega}{c}\dfrac{\partial E_z}{\partial y}\right) \\ B_y = \dfrac{i}{(\omega/c)^2 - k^2}\left(k\dfrac{\partial B_z}{\partial y} + \dfrac{\omega}{c}\dfrac{\partial E_z}{\partial x}\right) \end{cases} \tag{82}$$

We can plug these into the wave equation to obtain uncoupled equations for E_z and B_z:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^2 + \left(\frac{\omega}{c}\right)^2\right]\begin{pmatrix}E_z\\ B_z\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \tag{83}$$

In the context of waveguides, it's conventional to call the field in the direction of propagation the longitudinal ($\|$) field, whereas the fields perpendicular to it are called the transverse ($\perp$) fields. In this notation, we have

$$\boxed{\left[\nabla_\perp^2 - k^2 + \left(\frac{\omega}{c}\right)^2\right]\begin{pmatrix}E_\|\\ B_\|\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}} \tag{84}$$

Transverse Electric (TE) Modes: In TE modes, the **E**-field is purely in the $\perp$ plane, and $E_\| = 0$. At the surface of an ideal conductor, we have the boundary conditions $\mathbf{E}_\| = \mathbf{0}$ (the vector notation meaning that *all* components of **E** parallel to the walls must vanish at the walls, whereas the non-bolded $E_\|$ and $B_\|$ really just mean E_z and B_z), and the magnetic boundary conditions $B_\perp = 0$ at the walls. On the walls parallel to the xz plane, this means $B_y(y = 0) = B_y(y = b) = 0$, and $E_x(y = 0) = E_x(y = b) = 0$, and $E_z = 0$ throughout. This gives $\partial B_z/\partial y \equiv \partial B_\|/\partial y = 0$ at $y = 0$ and $y = a$. Another way of writing this is $\partial B_\parallel/\partial n = 0$ on the xz walls, where $\partial/\partial n \equiv \hat{\mathbf{n}} \cdot \nabla$. On the walls parallel to the yz plane, we have $B_x(x=0) = B_x(x=a) = 0$ and $E_y(x=0) = E_y(x=a) = 0$. This gives a similar boundary condition $\partial B_\parallel/\partial x \equiv \partial B_\parallel/\partial n = 0$ on the walls. Thus, for TE modes, we have

$$\boxed{\text{TE:} \quad \left(\nabla_\perp^2 - k^2 + (\omega/c)^2\right) B_\parallel = 0, \quad \text{with} \quad \partial B_\parallel/\partial n = 0 \ \text{ at the walls.}} \tag{85}$$

Transverse Magnetic (TM) Modes: In TM modes, the $\mathbf{B}$-field is purely in the $\perp$ plane, and $B_\parallel = 0$. Using the same general requirements for a perfect conductor that $\mathbf{E}_\parallel = \mathbf{0}$ and $B_\perp = 0$ at the walls, we see that this is actually all we need for a boundary condition, and we don't have to specify others:

$$\boxed{\text{TM:} \quad \left(\nabla_\perp^2 - k^2 + (\omega/c)^2\right) E_\parallel = 0, \quad \text{with} \quad \partial E_\parallel = 0 \ \text{ at the walls.}} \tag{86}$$

(b) We use the physicist's favorite way of solving partial differential equations: separation of variables.

TE modes: Take as an *ansatz*

$$B_z(x,y) = X(x)Y(y) \tag{87}$$

(we already know that $B_z \propto \exp[i(kz - \omega t)]$). We thus obtain

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\equiv -u_x^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\equiv -u_y^2} + (\omega/c)^2 - k^2 = 0 \tag{88}$$

for some positive constants u_x^2 and u_y^2, from which

$$\begin{cases} X''(x) = -u_x^2 \Rightarrow X(x) = C_1 \sin(u_x x) + C_2 \cos(u_x x) \\ Y'' = -u_y^2 \Rightarrow Y(y) = C_3 \sin(u_y y) + C_4 \cos(u_y y) \\ u_x^2 + u_y^2 = (\omega/c)^2 - k^2 \end{cases} \tag{89}$$

Our boundary conditions require that $B_x(x=0) = B_x(x=a) = 0$, and that $\partial B_z/\partial x = 0$ on those boundaries as well. (There are similar conditions for the other set of walls.) From this, we conclude that $u_x = m\pi/a$ and $u_y = n\pi/b$ for positive integers m, n, and thus our general TE solution, including the possibility of waves propagating in both directions, is

$$\boxed{\text{TE:} \quad B_z \equiv B_\parallel = \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\left(B_{0,+}e^{i(kz-\omega t)} + B_{0,-}e^{-i(kz+\omega t)}\right)} \tag{90}$$

The cutoff frequency ω_{mn} between propagating and exponentially-damped waves occurs where k becomes imaginary, and thus propagating solutions correspond to

$$\boxed{\text{TE:} \quad \omega > \omega_{mn} \equiv c\pi\sqrt{k^2 + \left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} \tag{91}$$

which allows us to write the dispersion relation $\boxed{\omega(k) = \sqrt{(ck)^2 + \omega_{m,n}^2}}$. Note that for TE modes, at least one of m, n must be nonzero. (If both were zero, $\omega = ck$ and our expressions (82) for $E_{x,y}$ and $B_{x,y}$ would be indeterminate. However, we could still go back to the general case (81), from which we would find that $B_z \equiv B_\parallel = \text{const}$. Applying Faraday's law would tell us that $B_\parallel = 0$, which contradicts the definition of a TE mode.)

TM modes: The solution for TM modes proceeds almost identically to the TE case, with

$$E_z(x,y) = X(x)Y(y), \tag{92}$$

and the same general solutions and constraints on $u_x = m\pi/a$ and $u_y = n\pi/b$. Applying the boundary conditions to force $E_\parallel = 0$ at the edges requires us to choose sines, so

$$\boxed{\text{TM:} \quad E_z \equiv E_\parallel = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\left(E_{0,+}e^{i(kz-\omega t)} + E_{0,-}e^{-i(kz+\omega t)}\right)} \tag{93}$$

In this case, *neither* m nor n can be zero, since this would cause $E_z \equiv E_\parallel = 0$ and hence we would have no TM mode at all. The dispersion relation and cutoff frequency are the same for TE and TM modes, subject to the caveat that both m, n must be nonzero in the TM modes.

(c) We can form our parallel-plate waveguide by just removing the walls at $y = 0$ and $y = b$. Since the planes are infinite in y, we only consider variations of the fields with x (remember, we already have the z dependence encoded in our complex exponential term). The boundary conditions remain the same as the previous part, except we no longer have the boundary conditions on the $y = 0$ and $y = b$ walls. The solutions for the TE and TM modes which satisfy the remaining boundary conditions are

$$\boxed{\text{TE:} \quad B_z \equiv B_{\parallel} = \cos\left(\frac{m\pi x}{a}\right)\left(B_{0,+}e^{i(kz-\omega t)} + B_{0,-}e^{-i(kz+\omega t)}\right)} \tag{94}$$

$$\boxed{\text{TM:} \quad E_z \equiv E_{\parallel} = \sin\left(\frac{m\pi x}{a}\right)\left(E_{0,+}e^{i(kz-\omega t)} + E_{0,-}e^{-i(kz+\omega t)}\right)} \tag{95}$$

which both share the cutoff frequency

$$\boxed{\omega > \omega_m \equiv c\pi\sqrt{k^2 + \left(\frac{m}{a}\right)^2}} \tag{96}$$

However, what happens if both $E_{\parallel}$ and $B_{\parallel}$ are zero (*i.e.*, $m = n = 0$ and hence $\omega = ck$)? Returning to (81), and using $E_y = \text{const}$ and $B_y = \text{const}$, we find

$$\begin{cases} -ikE_y = \dfrac{i\omega}{c}B_x, \quad -ikB_y = -\dfrac{i\omega}{c}E_x, \\ ikE_x = \dfrac{i\omega}{c}B_y, \quad ikB_x = -\dfrac{i\omega}{c}E_y \end{cases} \tag{97}$$

which is consistent with $\nabla_{\perp}^2\phi = 0$, where $\mathbf{E}_{\perp} = -\nabla_{\perp}\phi - \partial\mathbf{A}_{\perp}/\partial t$, where $\mathbf{A}$ is the magnetic vector potential. This tells us that $E_x = B_y$ and $E_y = B_x$. In the configuration in which $\mathbf{E} \propto \hat{\mathbf{x}}$ and $\mathbf{B} \propto \hat{\mathbf{y}}$, the perfect-conductor boundary conditions are satisfied automatically, and thus *the TEM mode propagates like a plane wave in free space*.

(d) Coming soon. The original solutions curiously stop after part **(a)**.

Quantum Mechanics Problem 1

(a) The Schrödinger equation with our *ansatz* $\psi(\mathbf{r}) = N\exp[-\beta r]$ for the $L = 0$ state is

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\exp[-\beta r] - \frac{e^2}{r}\exp[-\beta r] = E\exp[-\beta r] \tag{98}$$

$$-\frac{\hbar^2}{2m}\left(\beta^2 - \frac{2\beta}{r}\right) - \frac{e^2}{r} = E \tag{99}$$

This must hold for all values of r, so we require

$$\boxed{E = -\frac{\hbar^2\beta^2}{2m}, \quad \text{and} \quad \beta = \frac{me^2}{\hbar^2}} \tag{100}$$

(b) First, we need the normalization constant N. The electron must be found *somewhere*:

$$1 = \int |\psi|^2\, r^2 \sin\theta\, dr\, d\theta\, d\phi \tag{101}$$

$$= 4\pi N^2 \int_0^\infty r^2 \exp[-2\beta r]\, dr \tag{102}$$

$$= \frac{\pi N^2}{\beta^3} \tag{103}$$

Thus our ground-state wavefunction is

$$\psi(\mathbf{r}) = \sqrt{\frac{\beta^3}{\pi}}\exp[-\beta r]. \tag{104}$$

The decay of tritium (T) to ^{3}He increases the nuclear charge from e to $2e$, so $\beta_{\text{He}} = 2\beta_{\text{T}} \equiv 2\beta$. The ground-state wavefunctions for tritium and ^{3}He are

$$\psi_{\text{T}}(\mathbf{r}) = \sqrt{\frac{\beta^3}{\pi}}\exp[-\beta r] \tag{105}$$

$$\psi_{\text{He}}(\mathbf{r}) = \sqrt{\frac{8\beta^3}{\pi}}\exp[-2\beta r]. \tag{106}$$

The probability that the ^{3}He atom remains in its electronic ground state after the decay is given by the square of the overlap integral:

$$\text{Prob}(\text{T}(g.s.) \to {}^3\text{He}(g.s.)) = \left|\int \psi_{\text{T}}^\star \psi_{\text{He}}\, d^3\mathbf{r}\right|^2 \tag{107}$$

$$= \left|4\pi\sqrt{\frac{8\beta^6}{\pi^2}}\int_0^\infty e^{-3\beta r}\, dr\right|^2 \tag{108}$$

$$= \frac{8^3}{3^6} \tag{109}$$

$$\boxed{\text{Prob}(\text{T}(g.s.) \to {}^3\text{He}(g.s.)) \approx 0.7023} \tag{110}$$

(c) The angular momentum operator $\hat{\mathbf{L}}^2$ acts on the eigenstate $Y^\ell_{m_\ell}$ (*i.e*, a spherical harmonic with azimuthal quantum number ℓ and magnetic quantum number m_ℓ – I don't want to risk using m for both mass and the quantum number) as

$$\hat{\mathbf{L}}^2 Y^\ell_{m_\ell} = \hbar^2\ell(\ell+1)Y^\ell_{m_\ell} \tag{111}$$

Plugging this into the Schrödinger equation, we have (for $\ell = 1$)

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\right) r\exp[-\tilde{\beta}r] + \frac{\hbar^2(1+1)}{2mr^2} r\exp[-\tilde{\beta}r] - \frac{e^2}{r} r\exp[-\tilde{\beta}r] = Er\exp[-\tilde{\beta}r] \tag{112}$$

$$-\frac{\hbar^2}{2m}\left(-2\tilde{\beta} + r\tilde{\beta}^2 + \frac{2}{r}(1-\tilde{\beta}r)\right) + \frac{\hbar^2}{mr} - e^2 = Er \tag{113}$$

$$-\frac{\hbar^2}{2m}\left(-4\tilde{\beta} + r\tilde{\beta}^2\right) + \frac{\hbar^2}{2m} - \frac{\hbar^2}{2m} - e^2 = Er \tag{114}$$

The terms of order r^0 also have to vanish, since the right-hand side depends on r:

$$\frac{2\hbar^2\tilde{\beta}}{m} - e^2 = 0 \tag{115}$$

$$\boxed{\tilde{\beta} = \frac{me^2}{2\hbar^2}} \tag{116}$$

We are thus left with

$$\boxed{E = -\frac{\hbar^2\tilde{\beta}^2}{2m} = -\frac{me^4}{8\hbar^2}} \tag{117}$$

Quantum Mechanics Problem 2

(a) Expand the state $|\psi\rangle$ in terms of energy eigenstates $|a\rangle$ using the identity $\mathbf{1} = \sum_a |a\rangle \langle a|$:

$$|\psi\rangle = \sum_a \langle a|\psi\rangle |a\rangle \tag{118}$$

$$\langle\psi| \hat{H} |\psi\rangle = \sum_a \langle a|\psi\rangle \langle\psi| \hat{H} |a\rangle \tag{119}$$

$$= \sum_a E_a |\langle a|\psi\rangle|^2 \tag{120}$$

The probability of measuring the system in state $|a\rangle$ is therefore $p_a = |\langle a|\psi\rangle|^2$. We can see that $p_a \geq 0$ and that $\sum_a p_a = 1$. Thus we can interpret

$$\langle\psi| \hat{H} |\psi\rangle = \sum_a E_a p_a = \text{average value of } E. \tag{121}$$

Since the average is, by definition, greater than or equal to the lowest value (in this case, the ground state energy E_0), we have proved

$$\boxed{E_0 \leq \langle\psi| \hat{H} |\psi\rangle} \tag{122}$$

(b) First, we need to normalize the ground state $\psi(x) = N\sin(\pi x/L)$ on $0 \leq x \leq L$:

$$1 = \int_0^L |\psi|^2 \, dx \tag{123}$$

$$= N^2 \int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx \tag{124}$$

$$= \frac{N^2 L}{2} \tag{125}$$

Thus we have

$$\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) \tag{126}$$

Let $\hat{H} = \hat{H}_0 + \hat{V}(x)$, where $\hat{H}_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$. The ground state $|\psi\rangle$ is an eigenstate of $\hat{H}_0$ with energy

$$\langle\psi| \hat{H}_0 |\psi\rangle = \frac{\hbar^2\pi^2}{2mL^2} \tag{127}$$

In the variational method, we seek to minimize the value of $\langle\psi| \hat{H} |\psi\rangle$:

$$\langle\psi| \hat{H} |\psi\rangle = \langle\psi| \hat{H}_0 |\psi\rangle + \langle\psi| \hat{V} |\psi\rangle \tag{128}$$

$$= \frac{\hbar^2\pi^2}{2mL^2} + \frac{2V_0}{L}\int_a^L \sin^2\left(\frac{\pi x}{L}\right) dx \tag{129}$$

$$= \frac{\hbar^2\pi^2}{2mL^2} + \frac{2V_0}{\pi}\int_{\pi a/L}^{\pi} \sin^2 u \, du \tag{130}$$

Since we want to minimize $\langle\psi| \hat{H} |\psi\rangle$ with respect to L, we don't even need to evaluate the integral—we're just going to be differentiating it anyway! Let $K(u)$ be the antiderivative of $\sin^2 u$, *i.e.*, $K'(u) = \sin^2 u$:

$$0 = -\frac{\hbar^2\pi^2}{mL^3} + \frac{2V_0}{\pi}\frac{d}{dL}\left(K(\pi) - K\left(\frac{\pi a}{L}\right)\right) \tag{131}$$

$$= -\frac{\hbar^2\pi^2}{mL^3} - \frac{2V_0}{\pi}\sin^2\left(\frac{\pi a}{L}\right)\frac{d}{dL}\left(\frac{\pi a}{L}\right) \tag{132}$$

$$\frac{\hbar^2\pi^2}{mL} = 2aV_0 \sin^2\left(\frac{\pi a}{L}\right) \tag{133}$$

Defining $L \equiv a + \Delta$, with $\Delta/a \ll 1$, we have

$$\frac{\hbar^2\pi^2}{m(a+\Delta)} = 2aV_0 \sin^2\left(\frac{\pi a}{a+\Delta}\right) \tag{134}$$

$$= 2aV_0 \sin^2\left(\frac{\pi}{1+(\Delta/a)}\right) \tag{135}$$

$$\approx 2aV_0 \sin^2\left[\pi\left(1-\frac{\Delta}{a}\right)\right] \quad \text{since } (1+\epsilon)^n \approx 1+n\epsilon \text{ for } \epsilon \ll 1 \tag{136}$$

$$\frac{\hbar^2\pi^2}{2ma^2V_0[1+(\Delta/a)]} \approx \sin^2\left(\frac{\pi\Delta}{a}\right) \tag{137}$$

Up to this point, I am in agreement with the original written solution. However, the original written solution then finds

$$\frac{\hbar^2\pi^2}{2ma^2V_0} \approx \frac{\pi\Delta}{a}. \tag{138}$$

I do not know how they arrived at this. Here's what I get:

$$\frac{\hbar^2\pi^2}{2ma^2V_0}\left(1-\frac{\Delta}{a}+\frac{\Delta^2}{a^2}\right) \approx \frac{\pi^2\Delta^2}{a^2} \tag{139}$$

This gives the quadratic

$$\frac{\pi^2}{a^2}\left(1-\frac{\hbar^2}{2ma^2V_0}\right)\Delta^2 + \frac{\hbar^2\pi^2}{2ma^3V_0}\Delta - \frac{\hbar^2\pi^2}{2ma^2V_0} \approx 0 \tag{140}$$

I've attached the relevant pages of the original written solution at the end of this document, if you want to take a look. If you have any ideas for what's going on, please drop us a line at physREFS@mit.edu and we'll update the solutions. I don't believe this has anything to do with the physics (the last physics-motivated step was minimizing with respect to L several lines ago; from here on, we've just been trying to find a useful approximation for Δ); rather, I think it's just a math error in the original written solutions.

(d) Due to the disagreement between my result and the written solution's result for Δ in the previous part, I'm going to leave my final result for this part terms of Δ. The energy of the ground state is given by

$$\langle\psi|\,\hat{H}\,|\psi\rangle = \frac{\hbar^2\pi^2}{2mL^2} + \frac{2V_0}{\pi}\int_{\pi a/L}^{\pi} \sin^2 u \, du \tag{141}$$

$$= \frac{\hbar^2\pi^2}{2ma^2\,[1+(\Delta/a)]^2} + \text{the integral which contributes at } \mathcal{O}(\Delta^3/a^3) \tag{142}$$

Thus the ground state energy to $\mathcal{O}(\Delta/a)$ is

$$\boxed{\langle\psi|\,\hat{H}\,|\psi\rangle \approx \frac{\hbar^2\pi^2}{2ma^2}\left(1-\frac{2\Delta}{a}\right)} \tag{143}$$

which does agree with the written solution.

Statistical Mechanics Problem 1

(a) We model the bulk gas as N identical, non-interacting particles in a volume V, with single-particle energies

$$\epsilon = \frac{p_x^2 + p_y^2 + p_z^2}{2m} \tag{144}$$

The partition function Z is given by

$$Z(T,V,N) = \frac{1}{N! h^{3N}} \left[\int e^{-\epsilon(\mathbf{p})/kT} \, d^3\mathbf{p} \, d^3\mathbf{x} \right]^N \tag{145}$$

Note: I've always seen Z defined in terms of the Planck constant h and not the reduced Planck constant $\hbar$. The original written solution for this problem uses $\hbar$, but there is no indication of where the factors of (2π) went. I'm going to use h instead of $\hbar$, but this isn't a difference in the physics content of the problem.

$$\begin{aligned} Z(T,V,N) &= \frac{V^N}{N! h^{3N}} \left[\int_{-\infty}^{\infty} e^{-p_x^2/2mkT} \, dp_x \right]^{3N} && (146) \\ &= \frac{V^N}{N!} \left(\frac{2\pi mkT}{h^2} \right)^{3N/2} && (147) \end{aligned}$$

Use Stirling's approximation $N! \approx (N/e)^N$:

$$Z(T,V,N) \approx \left(\frac{eV}{N} \right)^N \left(\frac{2\pi mkT}{h^2} \right)^{3N/2} \tag{148}$$

Using the expression for the Helmholtz free energy $F = -kT \ln Z$, we find

$$F = -NkT \ln \left[\frac{eV}{N} \left(\frac{2\pi mkT}{h^2} \right)^{3/2} \right] \tag{149}$$

from which we can calculate the chemical potential

$$\begin{aligned} \mu = \left(\frac{\partial F}{\partial N} \right)_{T,V} &= -kT \ln \left[\frac{eV}{N} \left(\frac{2\pi mkT}{h^2} \right)^{3/2} \right] - NkT \frac{(-eV/N^2)(2\pi mkT/h^2)^{3/2}}{(eV/N)(2\pi mkT/h^2)^{3/2}} && (150) \\ &= -kT \ln \left[\frac{eV}{N} \left(\frac{2\pi mkT}{h^2} \right)^{3/2} \right] + kT && (151) \end{aligned}$$

Now we calculate the partition function of the surface gas, which is a two-dimensional ideal gas with single-particle energies

$$\epsilon = -E_0 + \frac{p_x^2 + p_y^2}{2m} \tag{152}$$

In terms of the area A, we have

$$\begin{aligned} Z(T,N,A) &= \frac{A^N}{N! h^{2N}} \left[e^{E_0/kT} \left(\int_{-\infty}^{\infty} e^{-p_x^2/2mkT} \, dp_x \right)^2 \right]^N && (153) \\ &= \frac{A^N}{N!} e^{NE_0/kT} \left(\frac{2\pi mkT}{h^2} \right)^N && (154) \\ &\approx \left(\frac{2\pi eAmkT}{Nh^2} \right)^N e^{NE_0/kT} && (155) \\ F(T,N,A) &= -kT \ln Z && (156) \\ &= -NkT \ln \left[\left(\frac{2\pi eAmkT}{Nh^2} \right) e^{E_0/kT} \right] && (157) \\ \mu &= \left(\frac{\partial F}{\partial N} \right)_{T,A}] && (158) \\ &= -kT \ln \left[\left(\frac{2\pi eAmkT}{Nh^2} \right) e^{E_0/kT} \right] + kT && (159) \end{aligned}$$

Now we equate $\mu_{\text{bulk}} = \mu_{\text{surface}}$, and set $N_{\text{bulk}}/V = n$ and $N_{\text{surface}}/A = \sigma$:

$$\frac{e(2\pi mkT)^{3/2}}{nh^3} = \frac{2\pi mekT}{\sigma h^2} e^{E_0/kT} \tag{160}$$

$$\boxed{\sigma(n,T) = \frac{hne^{E_0/kT}}{\sqrt{2\pi mkT}}} \tag{161}$$

This has the desired behavior that $\sigma(h \to 0) = 0$; hence, the physical adsorption is a manifestly quantum-mechanical effect.

Statistical Mechanics Problem 2

(a) The total number of bosons is given by integrating over the single-particle energy states:

$$N = \int_0^\infty \frac{D(\epsilon)}{e^{(\epsilon-\mu)/k_B T} - 1}\, d\epsilon, \tag{162}$$

where $D(\epsilon)$ is the density of states and μ is the chemical potential. The Bose factor in the integral also means that $\mu \leq 0$, or else the integral will be diverge, suggesting that $\mu = 0$ corresponds to some critical temperature T_c. We can model the individual atoms as free particles with energies

$$\epsilon = \frac{\hbar^2 k^2}{2m}. \tag{163}$$

The number of states # is given by the integral

$$\# = (2S+1)\int d^3\mathbf{x}\, \frac{d^3\mathbf{k}}{(2\pi)^3} \tag{164}$$

$$\frac{d\#}{dk} = D(k) = \frac{4\pi k V(2S+1)}{(2\pi)^3} \tag{165}$$

Changing variables to ϵ, we find

$$\frac{d\#}{d\epsilon} = D(\epsilon) = \frac{(2S+1)V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2} \tag{166}$$

As described before, the chemical potential $\mu(T_c) = 0$; otherwise, the Bose integral will diverge. Change variables to $x = \epsilon/k_B T_c$ so we can evaluate the integral:

$$N = \frac{(2S+1)V}{4\pi^2}\left(\frac{2mkT}{\hbar^2}\right)^{3/2}\underbrace{\int_0^\infty \frac{x^{1/2}}{e^x - 1}\, dx}_{\equiv I_c} \tag{167}$$

Defining $n \equiv N/V$, we find

$$\boxed{k_B T_c = \frac{\hbar^2}{2m}\left(\frac{4\pi^2 n}{I_c(2S+1)}\right)^{2/3}} \tag{168}$$

(b) For $T < T_c$, the Bose-Einstein distribution with $\mu = 0$ applies to all states except the ground state (since it has $\epsilon = 0$, having $\mu = 0$ results in a divergence in the integral). Thus we have to consider the number of particles in the ground state N_0 separately:

$$N = N_0(T) + \frac{(2S+1)VI_c}{4\pi^2}\left(\frac{2mkT}{\hbar^2}\right)^{3/2} \tag{169}$$

Using the result from the previous part, we can replace the coefficient on the second term:

$$N = N_0(T) + N\left(\frac{T}{T_c}\right)^{3/2} \tag{170}$$

Thus we find

$$\boxed{\frac{N_0(T)}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}} \tag{171}$$

As expected, all of the particles are in the ground state at $T = 0$, and none of them are in the ground state at $T = T_c$.

Quantum Mechanics II Solution

a) Expand $|\psi\rangle$ in the energy eigenstates

$$|\psi\rangle = \sum_a \langle E_a|\psi\rangle |E_a\rangle$$

$$\langle\psi|H|\psi\rangle = \sum_a E_a \left|\langle E_a|\psi\rangle\right|^2$$

$$p_a \equiv \left|\langle E_a|\psi\rangle\right|^2 \qquad \sum_a p_a = 1 \qquad p_a \geq 0$$

$$\langle\psi|H|\psi\rangle = \sum_a p_a E_a = \text{average } E$$

average is always bigger than smallest

$$E_0 \leq \langle\psi|H|\psi\rangle$$

b) $\psi(x) = N \sin(\pi x/L) \qquad 0 \leq x \leq L$

$$N^2 \int_0^L \sin^2 \frac{\pi x}{L}\, dx = N^2 \frac{L}{2} = 1 \qquad N = \sqrt{\frac{2}{L}}$$

$$\psi(x) = \sqrt{\frac{2}{L}} \sin(\pi x/L)$$

2

$$\langle\psi|H|\psi\rangle = \langle\psi|H_0|\psi\rangle + \langle\psi|V|\psi\rangle$$

ψ is an eigenstate of H_0

$$\langle\psi|H_0|\psi\rangle = \frac{\hbar^2}{2m}\frac{\pi^2}{L^2}$$

$$\langle\psi|V|\psi\rangle = \int_a^L V_0 \frac{2}{L}\sin^2\left(\frac{\pi x}{L}\right)dx$$

$$\frac{\pi x}{L} = y$$

$$\langle\psi|V|\psi\rangle = \frac{2V_0}{L}\int_{\frac{\pi a}{L}}^{\pi} \sin^2 y \,\frac{L\,dy}{\pi} = \frac{2V_0}{\pi}\int_{\frac{\pi a}{L}}^{\pi} \sin^2 y\, dy$$

$$\langle\psi|H|\psi\rangle = \frac{\hbar^2}{2m}\frac{\pi^2}{L^2} + \frac{2V_0}{\pi}\int_{\frac{\pi a}{L}}^{\pi} \sin^2 y\, dy$$

Take $\frac{d}{dL}$: $\quad -\frac{\hbar^2\pi^2}{mL^3} + \frac{2V_0}{\pi}\left(-\sin^2\frac{\pi a}{L}\right)\left(\frac{-\pi a}{L^2}\right) = 0$

$$\frac{\hbar^2\pi^2}{mL} = 2V_0\, a \sin^2\left(\frac{\pi a}{L}\right)$$

$$L = a + \Delta$$

$$\frac{\hbar^2\pi^2}{2mV_0 a(a+\Delta)} = \sin^2\left(\frac{\pi}{1+\Delta/a}\right)$$

$$\simeq \sin^2 \pi\left(1-\Delta/a\right)$$

$$\frac{\hbar^2\pi^2}{2mV_0 a^2} \simeq \frac{\pi\Delta}{a}$$

$$\Delta \simeq \frac{\hbar^2\pi}{2ma\,V_0}$$

$$\langle\psi|H|\psi\rangle = \frac{\hbar^2\pi^2}{2mL^2} + \frac{2V_0}{\pi}\int_{\frac{\pi a}{L}}^{\pi} \sin^2 y\, dy$$

$$\frac{\pi a}{L} = \frac{\pi}{1+\Delta/a} \simeq \pi\left(1-\Delta/a\right)$$

$\int$ is of order Δ^3

$$\frac{\hbar^2\pi^2}{2m(a+\Delta)^2} = \frac{\hbar^2\pi^2}{2ma^2}\,\frac{1}{(1+\Delta/a)^2}$$

$$\approx \frac{\hbar^2\pi^2}{2ma^2}\left(1-\frac{2\Delta}{a}\right)$$