Fall 2002 Exam: Solution to Electromagnetism Problem 1

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Consider the lowest *resonance mode of electromagnetic field in a cavity made of an ideal conductor. The cavity shape is a box $a \times b \times b$, $b < a$.*

a) *Find the resonance frequency and the spatial distribution of the fields $\vec{E}(r,t)$, $\vec{B}(r,t)$. Choose the mode with $\vec{E} \parallel \hat{y}$.*

Normal modes in a box are given by the product of standing waves in all three directions. The lowest resonant mode for an electric field in the $\hat{y}$ direction should not depend on y, so we can write the spatial dependence of the field as

$$\vec{E}(r) = E_0 \sin\left(\frac{\pi n_x}{a} x\right) \sin\left(\frac{\pi n_z}{b} z\right) \hat{y}. \tag{1}$$

Note that we use sine waves in the $\hat{x}$ and $\hat{z}$ directions since the solution must satisfy the boundary conditions $E_{\parallel} = 0$ at each wall. Then the frequency of each mode is given by

$$\omega = c\sqrt{k_x^2 + k_z^2} = c\sqrt{\left(\frac{\pi n_x}{a}\right)^2 + \left(\frac{\pi n_z}{b}\right)^2}. \tag{2}$$

We cannot set either n_x or n_z to zero, since from the boundary conditions we would have no standing wave at all! Therefore, the lowest resonant mode is given by $n_x = n_z = 1$, and the resonant frequency is given by

$$\boxed{\omega = c\pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}.} \tag{3}$$

Then, from equation (1), we have that the spatial distribution of the electric field is

$$\boxed{\vec{E}(r) = E_0 \sin\left(\frac{\pi}{a} x\right) \sin\left(\frac{\pi}{b} z\right) \hat{y}.} \tag{4}$$

The magnetic field can be found using Maxwell's equation $\nabla \times E = -dB/dt$.

$$-\frac{dB_x}{dt} = -\frac{dE_y}{dz}, \qquad -\frac{dB_z}{dt} = \frac{dE_y}{dx} \tag{5}$$

$$\boxed{B_x = -\frac{i\pi}{\omega b} E_0 \sin\left(\frac{\pi}{a} x\right) \cos\left(\frac{\pi}{b} z\right), \qquad B_z = \frac{i\pi}{\omega a} E_0 \cos\left(\frac{\pi}{a} x\right) \sin\left(\frac{\pi}{b} z\right)} \tag{6}$$

b) *Find the surface current and charge density on the cavity boundary. Make a schematic drawing.*

From the boundary conditions

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0} \tag{7}$$

$$B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K \tag{8}$$

we find that the charge densities on the $\hat{y}$ faces of the box are

$$\boxed{\sigma = \mp\epsilon_0 E_0 \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}z\right)e^{i\omega t}}. \tag{9}$$

The current density on the $\hat{x}$ sides is given by

$$\boxed{K_{x-sides} = -\frac{i\pi}{\omega a\mu_0}E_0\sin\left(\frac{\pi}{b}z\right)e^{i\omega t}\hat{y}.} \tag{10}$$

The current density on the $\hat{y}$ sides is given by

$$\boxed{K_{y-sides} = \mp\frac{i\pi E_0}{\omega\mu_0}e^{i\omega t}\left[\frac{1}{a}\cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}z\right)\hat{x} + \frac{1}{b}\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{b}z\right)\hat{z}\right].} \tag{11}$$

The current density on the $\hat{z}$ sides is given by

$$\boxed{K_{z-sides} = -\frac{i\pi}{\omega b\mu_0}E_0\sin\left(\frac{\pi}{a}x\right)e^{i\omega t}\hat{y}.} \tag{12}$$

c) *Find the electromagnetic force on the square sides $b \times b$. Indicate the direction of the force; show it on a drawing.*

The square sides of the box have no charge, and so are only subject to a magnetic force. In addition, the magnetic field at $x = 0$ and $x = a$ is purely in the $\hat{z}$ direction.

$$F_x = \int_0^b dy \int_0^b dz\, K_{x-sides} \times B_z(x = 0, a) \tag{13}$$

$$= -\int_0^b dy \int_0^b dz\, \frac{i\pi}{\omega a\mu_0}E_0\sin\left(\frac{\pi}{b}z\right)e^{i\omega t}\hat{y} \times \frac{i\pi}{\omega a}E_0\sin\left(\frac{\pi}{b}z\right)e^{i\omega t}(\pm\hat{z}) \tag{14}$$

$$= \frac{\pi^2 E_0^2}{\omega^2 a^2\mu_0}e^{i2\omega t}\left[\int_0^b dy\right]\left[\int_0^b dz\,\sin^2\left(\frac{\pi}{b}z\right)\right]\hat{y} \times (\pm\hat{z}) \tag{15}$$

$$= \frac{\pi^2 E_0^2}{\omega^2 a^2\mu_0}e^{i2\omega t}\,[b]\,[b/2]\,\hat{x} \tag{16}$$

$$\boxed{= \frac{\pi^2 b^2 E_0^2}{2\omega^2 a^2\mu_0}e^{i2\omega t}(\pm\hat{x})} \tag{17}$$

Thus, the forces on either side of the box are in opposite directions.