Consider the lowest resonance mode of electromagnetic field in a cavity made of an ideal conductor. The cavity shape is a box $a \times b \times b$, $b < a$.

**a) Find the resonance frequency and the spatial distribution of the fields $\vec{E}(r, t)$, $\vec{B}(r, t)$.** Choose the mode with $\vec{E} \parallel \hat{y}$.

Normal modes in a box are given by the product of standing waves in all three directions. The lowest resonant mode for an electric field in the $\hat{y}$ direction should not depend on $y$, so we can write the spatial dependence of the field as

$$\vec{E}(r) = E_0 \sin \left( \frac{\pi n_x}{a} x \right) \sin \left( \frac{\pi n_z}{b} z \right) \hat{y}.$$  

(1)

Note that we use sine waves in the $\hat{x}$ and $\hat{z}$ directions since the solution must satisfy the boundary conditions $E_\parallel = 0$ at each wall. Then the frequency of each mode is given by

$$\omega = c \sqrt{k_x^2 + k_z^2} = c \sqrt{\left( \frac{\pi n_x}{a} \right)^2 + \left( \frac{\pi n_z}{b} \right)^2}.$$  

(2)

We cannot set either $n_x$ or $n_z$ to zero, since from the boundary conditions we would have no standing wave at all! Therefore, the lowest resonant mode is given by $n_x = n_z = 1$, and the resonant frequency is given by

$$\omega = c \pi \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}.$$  

(3)

Then, from equation (1), we have that the spatial distribution of the electric field is

$$\vec{E}(r) = E_0 \sin \left( \frac{\pi}{a} x \right) \sin \left( \frac{\pi}{b} z \right) \hat{y}.$$  

(4)

The magnetic field can be found using Maxwell's equation $\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$.

$$-\frac{dB_x}{dt} = -\frac{dE_y}{dz}, \quad -\frac{dB_z}{dt} = \frac{dE_y}{dx}$$  

(5)

$$B_x = -\frac{i \pi}{\omega b} E_0 \sin \left( \frac{\pi}{a} x \right) \cos \left( \frac{\pi}{b} z \right), \quad B_z = \frac{i \pi}{\omega a} E_0 \cos \left( \frac{\pi}{a} x \right) \sin \left( \frac{\pi}{b} z \right)$$  

(6)

**b) Find the surface current and charge density on the cavity boundary.** Make a schematic drawing.

From the boundary conditions

$$E_\perp^{\text{above}} - E_\perp^{\text{below}} = \frac{\sigma}{\varepsilon_0}$$  

(7)

$$B_\parallel^{\text{above}} - B_\parallel^{\text{below}} = \mu_0 K$$  

(8)
we find that the charge densities on the $\hat{y}$ faces of the box are
\[ \sigma = \mp \epsilon_0 E_0 \sin \left( \frac{\pi}{a} x \right) \sin \left( \frac{\pi}{b} z \right) e^{i\omega t}. \] (9)

The current density on the $\hat{x}$ sides is given by
\[ K_{\hat{x}-\text{sides}} = -\frac{i\pi}{\omega \mu_0} E_0 \sin \left( \frac{\pi}{b} z \right) e^{i\omega t} \hat{y}. \] (10)

The current density on the $\hat{y}$ sides is given by
\[ K_{\hat{y}-\text{sides}} = \mp \frac{i\pi}{\omega \mu_0} E_0 \sin \left( \frac{\pi}{a} x \right) \left[ \frac{1}{a} \cos \left( \frac{\pi}{a} x \right) \sin \left( \frac{\pi}{b} z \right) \hat{x} + \frac{1}{b} \sin \left( \frac{\pi}{a} x \right) \cos \left( \frac{\pi}{b} z \right) \hat{z} \right]. \] (11)

The current density on the $\hat{z}$ sides is given by
\[ K_{\hat{z}-\text{sides}} = -\frac{i\pi}{\omega \mu_0} E_0 \sin \left( \frac{\pi}{a} x \right) e^{i\omega t} \hat{y}. \] (12)

c) Find the electromagnetic force on the square sides $b \times b$. Indicate the direction of the force; show it on a drawing.

The square sides of the box have no charge, and so are only subject to a magnetic force. In addition, the magnetic field at $x = 0$ and $x = a$ is purely in the $\hat{z}$ direction.

\[ F_x = \int_0^b dy \int_0^b dz \, K_{\hat{x}-\text{sides}} \times B_z(x = 0, a) \] (13)

\[ = -\int_0^b dy \int_0^b dz \, \frac{i\pi}{\omega \mu_0} E_0 \sin \left( \frac{\pi}{b} z \right) e^{i\omega t} \hat{y} \times \frac{i\pi}{\omega \mu_0} E_0 \sin \left( \frac{\pi}{b} z \right) e^{i\omega t} (\pm \hat{z}) \] (14)

\[ = \frac{\pi^2 E_0^2}{\omega^2 a^2 \mu_0} e^{i2\omega t} \left[ \int_0^b dy \right] \left[ \int_0^b dz \sin^2 \left( \frac{\pi}{b} z \right) \right] \hat{y} \times (\pm \hat{z}) \] (15)

\[ = \frac{\pi^2 E_0^2}{\omega^2 a^2 \mu_0} e^{i2\omega t} \left[ b \right] \left[ b/2 \right] \hat{x} \] (16)

\[ = \frac{\pi^2 b^2 E_0^2}{2\omega^2 a^2 \mu_0} e^{i2\omega t} (\pm \hat{x}) \] (17)

Thus, the forces on either side of the box are in opposite directions.