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DOCTORAL GENERAL EXAMINATION

PART II

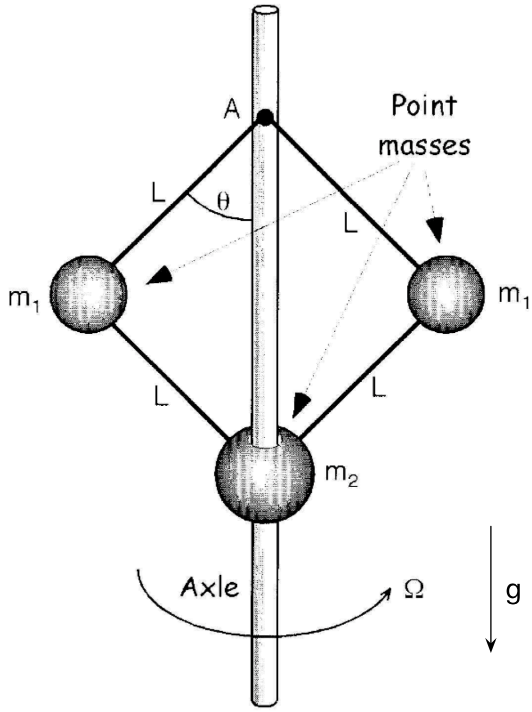
August 31, 2012

FIVE HOURS

1. This examination is divided into four sections, Mechanics, Electricity & Magnetism, Statistical Mechanics, and Quantum Mechanics, with two problems in each. Read both problems in each section carefully before making your choice. Submit ONLY one problem per section. IF YOU SUBMIT MORE THAN ONE PROBLEM FROM A SECTION, BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.
2. Use a separate booklet for each problem. Write your name and the problem number (I.2 for example) on the front of each booklet.
3. Calculators may not be used.
4. No books or reference materials may be used.

SECTION I: CLASSICAL MECHANICS

Classical Mechanics 1: Equilibrium States of a Governor



Consider the motion of an idealized "governor", shown in the above figure. This mechanical device consists of three **point** masses, two with mass m_1 and a third with mass m_2. The masses are connected by **massless** rigid rods of length L which are free to pivot about all joints. At a point A shown in the figure, the upper two rods are attached to a vertical axle. There is a downward gravitational acceleration g. As the angle θ between the rods and the axle varies, the mass m_2 slides freely along the axle.

For this problem, the axle is brought to a certain angular speed $\dot{\phi} = \Omega$ which is then kept constant in time. [*Aside*: In a real mechanical governor, the height of the mass m_2 regulates the fuel intake to an engine in order to keep Ω constant.]

(a) (*4 pts*) The system settles into an equilibrium configuration with angle θ_{eq}. Find the corresponding angular speed Ω.

(b) (*2 pts*) What is the minimum rotation speed Ω_{min} for which the equilibrium angle θ_{eq} is non-zero? What is the limiting behavior of θ_{eq} as $\Omega \to \infty$?

(c) (*4 pts*) For $\Omega > \Omega_{\text{min}}$, what is the frequency ω of small oscillations about θ_{eq}? Express your answer in terms of m_1, m_2, Ω, and θ_{eq}.

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(a) (*4 pts*) Since there are external torques on the system (to maintain a constant value of Ω), neither energy nor angular momentum is a constant of the motion. Let us therefore set up the Lagrangian for the problem. In terms of θ and ϕ and using the point A as the origin of the coordinate system, the location of one of the masses m_1 is

$$x_1 = L\sin\theta\cos\phi, \qquad y_1 = L\sin\theta\sin\phi, \qquad z_1 = -L\cos\theta, \tag{1}$$

and the location of mass m_2 is

$$x_2 = 0, \qquad y_2 = 0, \qquad z_2 = -2L\cos\theta. \tag{2}$$

The gravitational potential energy of the system is (including a factor of 2 for two m_1 masses):

$$U = 2m_1 g z_1 + m_2 g z_2 = -2gL(m_1 + m_2)\cos\theta. \tag{3}$$

The kinetic energy is:

$$K = 2\frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2\right) + \frac{m_2}{2}\dot{z}_2^2 = m_1 L^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) + 2m_2 L^2\sin^2\theta\,\dot{\theta}^2. \tag{4}$$

Using $\dot{\phi} = \Omega$, the Lagrangian $\mathcal{L} = K - U$ is:

$$\mathcal{L} = L^2(m_1 + 2m_2\sin^2\theta)\dot{\theta}^2 + m_1 L^2\Omega^2\sin^2\theta + 2gL(m_1 + m_2)\cos\theta. \tag{5}$$

The problem has been reduced to a single variable θ. The Lagrange equation of motion ($\frac{\partial\mathcal{L}}{\partial\theta} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = 0$) yields

$$-L^2(m_1+2m_2\sin^2\theta)\ddot{\theta} - 2m_2L^2\sin\theta\cos\theta\,\dot{\theta}^2 + m_1L^2\Omega^2\sin\theta\cos\theta - gL(m_1+m_2)\sin\theta = 0. \tag{6}$$

In equilibrium, we have $\ddot{\theta} = \dot{\theta} = 0$, yielding

$$m_1L^2\Omega^2\sin\theta_{\rm eq}\cos\theta_{\rm eq} - gL(m_1 + m_2)\sin\theta_{\rm eq} = 0. \tag{7}$$

Solving for Ω, we obtain the desired relation between the equilibrium angle $\theta_{\rm eq}$ and the rotation frequency of the axle:

$$\boxed{\Omega = \sqrt{\frac{g}{L}\frac{m_1 + m_2}{m_1}\frac{1}{\cos\theta_{\rm eq}}}.} \tag{8}$$

(b) (*2 pts*) Since $\cos\theta \le 1$, the minimum rotation speed $\Omega_{\rm min}$ is

$$\boxed{\Omega_{\rm min} = \sqrt{\frac{g}{L}\frac{m_1 + m_2}{m_1}}.} \tag{9}$$

As $\Omega \to \infty$, we have $\cos\theta \to 0$, so $\boxed{\theta_{\rm eq} \to \pi/2}$.

(c) (*4 pts*) For small oscillations around θ_{eq}, we can neglect the $\dot{\theta}^2$ term in the equation of motion since it is second order in the smallness, compared to first order for the $\ddot{\theta}$ term. This gives the approximate equation of motion

$$(m_1 + 2m_2 \sin^2\theta)\ddot{\theta} = m_1\Omega^2 \sin\theta\cos\theta - \frac{g}{L}(m_1 + m_2)\sin\theta. \tag{10}$$

We can simplify this expression a bit by relating $\frac{g}{L}(m_1 + m_2)$ to the equilibrium angle θ_{eq}.

$$(m_1 + 2m_2 \sin^2\theta)\ddot{\theta} = m_1\Omega^2 \sin\theta(\cos\theta - \cos\theta_{\text{eq}}). \tag{11}$$

Keeping only the first order term in the smallness, we can replace

$$\sin\theta(\cos\theta - \cos\theta_{\text{eq}}) \to \sin\theta_{\text{eq}}\left(-\sin\theta_{\text{eq}}(\theta - \theta_{\text{eq}})\right). \tag{12}$$

This leads to the equation of motion for small oscillations

$$(m_1 + 2m_2 \sin^2\theta_{\text{eq}})\ddot{\theta} = -m_1\Omega^2 \sin^2\theta_{\text{eq}}(\theta - \theta_{\text{eq}}). \tag{13}$$

From this we find that the equation describing small oscillations about θ_{eq} is just that of a harmonic oscillator:

$$\ddot{\theta} = -\omega^2(\theta - \theta_{\text{eq}}), \tag{14}$$

with frequency

$$\boxed{\omega^2 = \Omega^2 \frac{m_1 \sin^2\theta_{\text{eq}}}{m_1 + 2m_2 \sin^2\theta_{\text{eq}}}.} \tag{15}$$

Classical Mechanics 2: Relativistic spaceship

A spaceship ejects fuel with a constant exhaust velocity $v_{\rm ex}$ in its own frame, and travels in a straight line through empty space with negligible gravity. Initially the spaceship has a full fuel tank and a total rest mass M_0 (spaceship + fuel).

(a) (*2 pts*) Let S' be an inertial frame of reference in which the spaceship is instantaneously at rest, and let $M(t')$ denote the mass of the spaceship as a function of the time t' in this frame. Derive an expression for $a'(t')$, the acceleration of the spaceship in the frame S', at the time t' when it is at rest. The expression may involve $M(t')$ and any of its derivatives. Assume that the exhaust speed is non-relativistic: $v_{\rm ex} \ll c$.

(b) (*3 pts*) The spaceship starts from rest in an inertial frame S and accelerates in the x-direction. The engines fire until all the fuel is consumed. With an empty fuel tank, the rest mass of the spaceship is M_0/f where $f > 1$ is a constant.

Assume that the spaceship achieves a **non-relativistic** final speed in frame S. Find this final speed and write your answer in terms of $v_{\rm ex}$ and f.

(c) (*5 pts*) Suppose the same conditions hold as in part (b), except that the spaceship achieves a **relativistic** speed in frame S. The exhaust velocity $v_{\rm ex}$ is still assumed to be non-relativistic. Find the final speed of the rocket in S in terms of $v_{\rm ex}$ and f, using special-relativistic kinematics and dynamics. Hint: you may want to relate the accelerations a' and a that are observed in frames S' and S respectively.

Possibly useful integrals:

$$\int \frac{ds}{(1-s^2)^{1/2}} = \sin^{-1}(s), \tag{1}$$

$$\int \frac{ds}{(1-s^2)} = \frac{1}{2}\ln\frac{1+s}{1-s}, \tag{2}$$

$$\int \frac{ds}{(1-s^2)^{3/2}} = \frac{s}{(1-s^2)^{1/2}}. \tag{3}$$

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(a) Let dM be the change in mass of the spaceship during the time interval dt'. Since the spaceship is ejecting fuel, $dM < 0$. Let dv' be the corresponding change in the spaceship's velocity. From the conservation of (non-relativistic) momentum,

$$(-dM)\ (-v_{\rm ex}) + (M + dM)dv' = 0. \tag{4}$$

Ignoring the second-order term $dM\ dv'$,

$$dv' = -v_{\rm ex}\frac{dM}{M} \tag{5}$$

$$a' = \frac{dv'}{dt'} = -v_{\rm ex}\frac{1}{M}\frac{dM}{dt'}. \tag{6}$$

(b) Since the motion is nonrelativistic in both S and S', we have both $a' = a$ and $dt' = dt$, implying $dv' = dv$. We may therefore integrate the expression for dv' derived above to obtain the final velocity:

$$dv = dv' = -v_{\text{ex}}\frac{dM}{M} \tag{7}$$

$$v_{\text{final}} = -v_{\text{ex}}\int_{M_0}^{M_0/f}\frac{dM}{M} \tag{8}$$

$$v_{\text{final}} = -v_{\text{ex}}\ln(1/f) = v_{\text{ex}}\ln f. \tag{9}$$

(c) There are at least two ways to solve this problem.

One way is to relate a' and a. In this case, where the acceleration is along the direction of motion, and the spaceship is at rest in S', the relation is simple:

$$a' = \gamma^3 a, \tag{10}$$

as we will now derive. (Strictly speaking the student need not derive this formula, although it seems unlikely that this would be memorized.)

For the derivation, let v be the instantaneous velocity of the rocket seen in S. Align the x-axis with the direction of motion. The Lorentz transformations for differential displacements are

$$dx = \gamma(dx' + v\,dt') \tag{11}$$

$$dt = \gamma\left(dt' + \frac{v}{c^2}dx'\right) \tag{12}$$

$$\tag{13}$$

We use these equations to relate the velocity v_x of an object seen in S to the velocity v'_x seen in S':

$$v_x = \frac{dx}{dt} = \frac{dx' + v\,dt'}{dt' + \frac{v}{c^2}dx'} \tag{14}$$

$$= \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} \tag{15}$$

$$= \frac{v'_x + v}{1 + \frac{vv'_x}{c^2}} \tag{16}$$

This is the familiar velocity addition formula (which could also have been taken from memory as a starting point for this derivation). To relate the accelerations a_x and a'_x we need to take the differential of this formula,

$$dv_x = \frac{dv'_x}{1 + \frac{vv'_x}{c^2}} - \frac{(v'_x + v)dv'_x}{(1 + \frac{vv'_x}{c^2})^2} \tag{17}$$

$$= \frac{dv'_x}{(1 + \frac{vv'_x}{c^2})^2}\left[1 + \frac{v'_x v}{c^2} - \frac{v'_x v}{c^2} - \frac{v^2}{c^2}\right] \tag{18}$$

$$= dv'_x\,\frac{1 - \frac{v^2}{c^2}}{(1 + \frac{vv'_x}{c^2})^2}. \tag{19}$$

We now use the fact that $v'_x = 0$, giving

$$dv_x = dv'_x \left(1 - \frac{v^2}{c^2}\right) = \frac{dv'_x}{\gamma^2}. \tag{20}$$

To relate the accelerations of the spaceship, we divide by dt, and use the fact that $dt = \gamma dt'$ since $dx' = 0$,

$$a_x = \frac{dv_x}{dt} = \frac{dv'_x}{\gamma^2}\frac{1}{\gamma dt'} = \frac{1}{\gamma^3}\frac{dv'_x}{dt'} = \frac{a'_x}{\gamma^3} \tag{21}$$

as advertised.

We can now proceed as we did in part (b) but with the corrected expression for the acceleration:

$$\frac{dv}{dt} = \frac{1}{\gamma^3}\left(-v_{\rm ex}\frac{1}{M}\frac{dM}{dt'}\right) \tag{22}$$

$$dv = \frac{1}{\gamma^3}\left(-v_{\rm ex}\frac{dM}{M}\frac{dt}{dt'}\right) \tag{23}$$

$$dv = \frac{1}{\gamma^2}\left(-v_{\rm ex}\frac{dM}{M}\right) \tag{24}$$

where in the last step we used $dt = \gamma dt'$ again. To perform the integral and obtain the final velocity we must bring the velocity-dependent term γ^2 onto the left side,

$$\gamma^2 dv = \frac{dv}{1 - \frac{v^2}{c^2}} = -v_{\rm ex}\frac{dM}{M} \tag{25}$$

$$\int_0^{v_{\rm final}} \frac{dv}{1 - \frac{v^2}{c^2}} = -v_{\rm ex}\int_{M_0}^{M_0/f}\frac{dM}{M}. \tag{26}$$

Denoting $\beta \equiv v/c$, and $\beta_{\rm ex} \equiv v_{\rm ex}/c$,

$$\int_0^{\beta_{\rm final}} \frac{c\,d\beta}{1 - \beta^2} = c\,\beta_{\rm ex}\ln f \tag{27}$$

$$\frac{1}{2}\ln\left(\frac{1+\beta_{\rm final}}{1-\beta_{\rm final}}\right) = \beta_{\rm ex}\ln f \tag{28}$$

$$\left(\frac{1+\beta_{\rm final}}{1-\beta_{\rm final}}\right) = f^{2\beta_{\rm ex}} \tag{29}$$

$$\beta_{\rm final} = \frac{f^{2\beta_{\rm ex}} - 1}{f^{2\beta_{\rm ex}} + 1} \tag{30}$$

$$v_{\rm final} = c\,\frac{f^{2v_{\rm ex}/c} - 1}{f^{2v_{\rm ex}/c} + 1}. \tag{31}$$

A second way to solve this problem relies on the concept of *rapidity* r, defined by $v/c = \tanh(r)$. This is a simplifying concept because for motions along a line, rapidities are additive and changes in rapidity are frame-independent. This is unlike the case for velocities and changes in velocity.

For small velocities, $r \approx v/c$, and therefore

$$dr = dr' = dv'/c = -\frac{v_{\text{ex}}}{c}\frac{dM}{M} \tag{32}$$

We may integrate this equation to find the final rapidity,

$$r_{\text{final}} = -\frac{v_{\text{ex}}}{c}\ln f, \tag{33}$$

which can be expressed in terms of velocity as

$$v_{\text{final}} = c\ \tanh\left(\frac{v_{\text{ex}}}{c}\ln f\right), \tag{34}$$

which is equivalent to the answer derived previously.

SECTION II: ELECTRICITY & MAGNETISM

Electromagnetism 1: Magnetic monopoles

Imagine there exist magnetic monopoles. The magnetic charge density ρ_m and magnetic current density $\vec{J}_m$ are analogous to the electric charge density ρ and electric current density $\vec{J}$. In particular, magnetic charge is conserved locally, just as electric charge is conserved. Maxwell's equations (in Gaussian units) are

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho, \quad \vec{\nabla} \cdot \vec{B} = 4\pi\rho_m, \quad \vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}, \quad \vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} - \frac{4\pi}{c}\vec{J}_m. \tag{1}$$

(a) (*1 pt*) Consider a pointlike magnetic monopole with magnetic charge g. In quantum mechanics, the requirement that the wavefunction phase is single-valued leads to a constraint on the magnetic flux through any closed surface,

$$\frac{e}{\hbar c} \oint \vec{B} \cdot d\vec{S} = 2\pi n, \tag{2}$$

where n is an integer and $e > 0$ is the magnitude of the electron charge. From this condition, it follows that magnetic charge must be quantized, i.e., all magnetic charges are integral multiples of some elementary unit. Derive the elementary unit of magnetic charge in terms of fundamental constants.

(b) (*2 pts*) Consider a system of two point charges: a positron with charge e and a magnetic point charge g separated by a distance $2d$. Use the cylindrical coordinate system illustrated in Figure 1 (on the next page), in which the origin is at the midpoint between the charges, the positron is at $z = -d$ and the magnetic charge is at $z = +d$. Compute the electric and magnetic fields as a function of (r, θ, z).

(c) (*6 pts*) Compute the total angular momentum contained in the fields from part (c).

The following integrals may be useful:

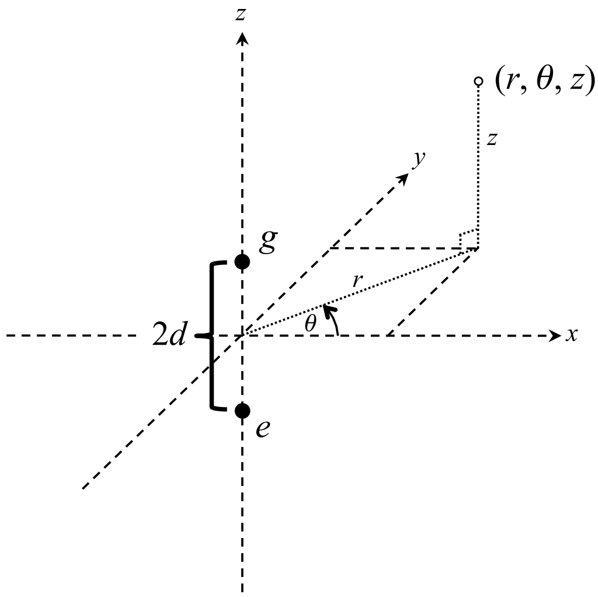
$$\int \frac{x\,dx}{(x^2 + \beta x + \gamma)^{3/2}} = \left(\frac{1}{\beta^2/4 - \gamma}\right) \frac{\beta x/2 + \gamma}{(x^2 + \beta x + \gamma)^{1/2}} \tag{3}$$

$$\int_{-\infty}^{\infty} dx\, \frac{(a^2 + x^2) - \sqrt{(a^2 - x^2)}}{(a^2 + x^2)^2 - (a^2 - x^2)^2} = \frac{2}{a} \tag{4}$$

The following vector identity may be useful:

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}). \tag{5}$$

(d) (*1 pt*) Use the results from part (b) and (d), show that the angular momentum is also quantized and derive the elementary unit of angular momentum.



(a) The integral form of the new magnetic divergence law is

$$\oint \vec{B} \cdot d\vec{S} = 4\pi g_{\rm tot}, \tag{6}$$

where $g_{\rm tot}$ is the total magnetic charge enclosed by the surface over which the integral is evaluated. Applying this law to a surface surrounding an isolated point magnetic charge g, and using the quantization condition provided,

$$\oint \vec{B} \cdot d\vec{S} = 4\pi g = 2\pi n \, \frac{\hbar c}{e}, \tag{7}$$

and therefore the monopole charge must be quantized as

$$g = \frac{\hbar c}{2e} \, n, \tag{8}$$

and the elementary unit of magnetic charge is $\hbar c/2e$.

(b) Using Coulomb's law, and the analogous law for magnetic monopoles, each charge produces a field varying as $\vec{s}/s^3$ where $\vec{s} = r\hat{r} + (z \pm d)\hat{z}$ is the vector from the charge to the location (r, z, θ). More explicitly,

$$\vec{E} = e \, \frac{r\hat{r} + (z+d)\hat{z}}{[r^2 + (z+d)^2]^{3/2}}, \tag{9}$$

$$\vec{B} = g \, \frac{r\hat{r} + (z-d)\hat{z}}{[r^2 + (z-d)^2]^{3/2}}. \tag{10}$$

(c) The momentum density of the electromagnetic fields is $\vec{S}/c^2$, where $\vec{S}$ is the Poynting vector field,

$$\vec{S} = \frac{c}{4\pi}\left(\vec{E} \times \vec{B}\right), \tag{11}$$

and the angular momentum density $\vec{\ell}$ is

$$\begin{aligned}
\vec{\ell} &= \vec{r} \times \frac{\vec{S}}{c^2} && (12)\\
&= \frac{1}{4\pi c}\,\vec{r} \times \left(\vec{E} \times \vec{B}\right) && (13)\\
&= \frac{1}{4\pi c}\left[\vec{E}(\hat{r}\cdot\vec{B}) - \vec{B}(\hat{r}\cdot\vec{E})\right]. && (14)
\end{aligned}$$

The expression in square brackets is

$$\frac{e[r\hat{r} + (z+d)\hat{z}]gr - g[r\hat{r} + (z-d)\hat{z}]er}{[r^2 + (z+d)^2]^{3/2}\,[r^2 + (z-d)^2]^{3/2}} = \frac{2egrd\ \hat{z}}{[r^2 + (z+d)^2]^{3/2}\,[r^2 + (z-d)^2]^{3/2}}. \tag{15}$$

Thus we see the angular momentum is in the $\hat{z}$ direction. Integrating ℓ_z over all space,

$$\begin{aligned}
L &= \int_{-\infty}^{\infty} dz \int_0^{\infty} 2\pi r\ dr\ \frac{1}{4\pi c}\frac{2egrd}{[r^4 + d^4 + z^4 - 2z^2d^2 + 2r^2z^2 + 2r^2d^2]^{3/2}}. && (16)\\
&= \frac{egd}{2c}\int_{-\infty}^{\infty} dz \int_0^{\infty} \frac{2r^3\ dr}{[r^4 + d^4 + z^4 - 2z^2d^2 + 2r^2z^2 + 2r^2d^2]^{3/2}}. && (17)
\end{aligned}$$

With the substitution $x = r^2$ this integral becomes

$$\begin{aligned}
L &= \frac{egd}{2c}\int_{-\infty}^{+\infty} dz \int_0^{\infty} \frac{x\ dx}{[x^2 + (2z^2 + 2d^2)x + (d^4 + z^4 - 2z^2d^2)]^{3/2}} && (18)\\
&= \frac{egd}{2c}\int_{-\infty}^{+\infty} dz \int_0^{\infty} dx\ \frac{x\ dx}{[x^2 + \beta x + \gamma]^{3/2}} && (19)
\end{aligned}$$

where $\beta \equiv 2(z^2 + d^2)$ and $\gamma \equiv (d^2 - z^2)^2$. Using the first integral provided,

$$\begin{aligned}
L &= \frac{egd}{2c}\int_{-\infty}^{+\infty} dz \left(\frac{1}{\beta^2/4 - \gamma}\right) \left.\frac{\gamma + \beta x/2}{(x^2 + \beta x + \gamma)^{1/2}}\right|_0^{\infty} && (20)\\
&= \frac{egd}{2c}\int_{-\infty}^{+\infty} dz\ \frac{\beta/2 - \gamma^{1/2}}{\beta^2/4 - \gamma} && (21)\\
&= \frac{egd}{2c}\int_{-\infty}^{+\infty} dz\ \frac{d^2 + z^2 - \sqrt{(d^2 - z^2)^2}}{(d^2 + z^2)^2 - (d^2 - z^2)^2}. && (22)
\end{aligned}$$

Using the second integral provided,

$$L = \frac{egd}{2c}\frac{2}{d} = \frac{eg}{c}. \tag{23}$$

(d) Using the results from parts (b) and (d),

$$L = \frac{eg}{c} = \frac{e\hbar c}{2e}\,n = \frac{\hbar}{2}\,n. \tag{24}$$

Electromagnetism 2: Eddy current and decaying magnetic field

An infinitely long, thin-walled cylinder has a circular cross-section with inner radius a and outer radius b. Let the z-axis be coincident with the symmetry axis of the cylinder, and let r be the *cylindrical* radial coordinate, i.e., the distance perpendicular to the z-axis.

The cylinder is made of a non-magnetic metal with electrical conductivity σ. For times $t < 0$ the system has come to equilibrium with an applied external magnetic field. As a result the magnetic field is uniform and constant throughout all space, $\vec{B} = B_c\,\hat{z}$, and the current density is everywhere zero, $\vec{J} = 0$.

At $t = 0$ the sources responsible for the applied magnetic field are turned off. Subsequently, a current develops in the metal which supports a residual magnetic field $\vec{B}(r,t)$ in the z-direction. The residual magnetic field inside the cylinder begins with the constant value B_c and eventually relaxes to zero.

In what follows you should assume the wall of the cylinder is sufficiently thin that the fields and currents do not vary significantly between $r = a$ and $r = b$.

(a) (*3 pts*) A zeroth-order approximation assumes that the residual magnetic field is independent of position inside the cylinder, and zero elsewhere:

$$\vec{B}_0 = \begin{cases} B_0(t)\hat{z} & r \le a \\ 0 & r > a \end{cases} \tag{1}$$

Derive a differential equation for $B_0(t)$ under these assumptions, and give the relevant solution.

(b) (*4 pts*) Calculate the first r-dependent correction $\vec{B}_1 = B_1(r,t)\hat{z}$ to the residual magnetic field, by finding the electric field $\vec{E}(r,t)$ that is induced by $\vec{B}_0(t)$, and then the magnetic field that is induced by $\vec{E}(r,t)$. You may assume that the correction $\vec{B}_1(r,t)$ vanishes at $r = 0$. You may also express $\vec{B}_1$ in terms of $\vec{B}_0$.

(c) (*3 pts*) What dimensionless parameter must be $\ll 1$ in order to ensure that $\vec{B}_1$ is a small correction? Express this inequality in terms of the time required for light to cross the cylinder.

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(a) A self-consistent equation for $B_0(t)$ is derived as follows. First, the disappearing magnetic field will produce a circulating electric field within the walls of the cylinder, via Faraday's law. Applying Faraday's law to a circular path with a radius a:

$$\begin{aligned} \oint \vec{E}\cdot d\vec{l} &= -\frac{d}{dt}\int_A \vec{B}\cdot d\vec{A}, \\ 2\pi a E_0 &= -\pi a^2 \frac{dB_0}{dt}, \end{aligned} \tag{2}$$

where E_0 is the magnitude of the electric field within the walls.

Next, using Ohm's law, the circulating current density J within the walls is

$$J = \sigma E_0. \tag{3}$$

Finally, the current density J is related to the residual magnetic field $\vec{B}$ through Ampere's law. We ignore displacement current, which would give rise to an r-dependence in the magnetic field. Thus we have a situation similar to an idealized solenoid, for which the magnetic field is $\mu n I$, with n the number of turns per unit length and I the current per turn. Here, the total current per unit length is $J(b-a)$ and hence we replace $nI \to J(b-a)$ giving

$$B_0 = \mu_0 J(b-a). \tag{4}$$

Putting together Eqns.(2-4),

$$\begin{aligned} 2\pi a \ \frac{J}{\sigma} &= -\pi a^2 \frac{dB_0}{dt}, & (5)\\ 2\pi a \ \frac{B_0}{\mu_0 (b-a)\sigma} &= -\pi a^2 \frac{dB_0}{dt}, & (6)\\ \frac{dB_0}{dt} &= -\frac{2}{\mu_0 a(b-a)\sigma} \ B_0. & (7) \end{aligned}$$

The relevant solution is exponential decay from the initial value,

$$B_0(t) = B_c \exp(-t/\tau) \quad \text{with} \ \ \tau \equiv \frac{\mu_0 a(b-a)\sigma}{2}. \tag{8}$$

(b) The origin of the first-order correction is the displacement current created by the time-varying electric field within the cylinder. The electric field circulates around the $\hat{z}$ axis, i.e., $\vec{E} = E_\phi(r,t)\hat{\phi}$. Apply Faraday's law to a circle of radius $r < a$:

$$2\pi r \ E_\phi(r,t) = -\pi r^2 \frac{dB_0}{dt} \quad \to \quad E_\phi(r,t) = -\frac{r}{2}\frac{dB_0}{dt}. \tag{9}$$

Next apply Ampere's law to a rectangle with vertical sides of length h, with one vertical side along the z-axis and the other at cylindrical radius r. Since there is no current density within the cylinder, we have only the displacement current,

$$\begin{aligned} \oint \vec{B}\cdot d\vec{l} &= \mu_0\epsilon_0 \int_A \frac{\partial \vec{E}}{\partial t}\cdot d\vec{A} = & (10)\\ hB_1(r,t) &= \mu_0\epsilon_0 \int h \ dr' \left(-\frac{r'}{2}\frac{d^2B_0}{dt^2}\right) & (11)\\ B_1(r,t) &= -\frac{\mu_0\epsilon_0}{2}\,\frac{B_0}{\tau^2} \int_0^r r' dr' & (12) \end{aligned}$$

and, using $\mu_0\epsilon_0 = 1/c^2$,

$$B_1(r,t) = -\left(\frac{r}{2c\tau}\right)^2 B_0(t). \tag{13}$$

This is the first-order r-dependent correction to the residual magnetic field.

(c) The factor $r/2c\tau$ must be small for all $r < a$. Hence the desired condition is

$$\frac{a}{2c\tau} \ll 1, \tag{14}$$

or equivalently the light-crossing time $2a/c$ must be much shorter than 4τ. It is fine to give the order-of-magnitude inequality $a/c \ll \tau$.

SECTION III: STATISTICAL MECHANICS

Statistical Mechanics 1: Cooling by demagnetization

Consider an insulating crystal formed by atoms of mass m. Each atom carries a nuclear spin of $\hbar/2$ but no net electron spin. The number density of the atoms is n. The crystal supports sound waves (phonons) with two transverse and one longitudinal mode, all with the same velocity v.

(a) (*4 pts*) First assume that the nuclear spin is completely polarized by a strong magnetic field. Find the free energy per unit volume $f(T)$ of the crystal, where the temperature of the crystal is T. (Your answer may contain a *dimensionless* integral which you need not evaluate.)

(b) (*4 pts*) Initially, the nuclear spin is completely polarized by a strong magnetic field, and the temperature of the crystal is T. Now we slowly decrease the magnetic field to zero while keeping the system isolated from its surroundings. Find the final temperature T' of the crystal after the demagnetization. Here we assume that there is sufficient coupling between the phonon and the spin degrees of freedom that the system remains in thermal equilibrium throughout the demagnetization process. However, the coupling has a negligible effect on the free energy of the system.

(c) (*2 pts*) Following your analysis of part b), are there circumstances in which the crystal remains magnetized even after the external magnetic field is reduced to zero? Explain your result.

SOLUTION:

(a) The free energy of a boson system is

$$F = k_{\rm B}T \sum_k \ln(1 - e^{-\beta(E_k-\mu)}) \tag{1}$$

For phonons $\mu = 0$ and $E_k = vk\hbar$. We have three sound modes, two transverse, one longitudinal. Thus,

$$\begin{aligned} f &= 3k_{\rm B}T \int \frac{{\rm d}^3k}{(2\pi)^3} \ln(1 - e^{-\beta v\hbar k}) \\ &= \frac{3k_{\rm B}T}{(\beta v\hbar)^3} \int_0^\infty \frac{{\rm d}x}{2\pi^2}\, x^2 \ln(1-e^{-x}) \\ &= -\frac{3k_{\rm B}^4T^4}{(v\hbar)^3} I, \end{aligned} \tag{2}$$

where

$$I = -\int_0^\infty \frac{{\rm d}x}{2\pi^2}\, x^2 \ln(1-e^{-x}) > 0 \tag{3}$$

(b) The entropy is conserved during the demagnetization process. The change in the entropy density from the nuclear spins is

$$\Delta s_{\rm sp} = k_{\rm B} n \ln 2 \tag{4}$$

The entropy density for the phonons is

$$s_{\rm ph} = -\frac{\partial f}{\partial T} = \frac{12k_{\rm B}^4T^3}{(v\hbar)^3} I \tag{5}$$

Thus

$$(T^3 - T'^3)\frac{12k_B^4}{(v\hbar)^3} I = k_{\rm B} n \log 2. \tag{6}$$

We see that

$$T' = \left(\frac{\frac{12k_{\rm B}^3T^3}{(v\hbar)^3} I - n\log 2}{\frac{12k_{\rm B}^3}{(v\hbar)^3} I} \right)^{1/3}. \tag{7}$$

when

$$T > \left(\frac{n(v\hbar)^3 \log 2}{12k_{\rm B}^3 I} \right)^{1/3} \tag{8}$$

and

$$T' = 0 \tag{9}$$

when

$$T < \left(\frac{n(v\hbar)^3 \log 2}{12k_{\rm B}^3 I} \right)^{1/3}. \tag{10}$$

(c) When $T < \left(\frac{n(v\hbar)^3 \log 2}{12 k_B^3 I}\right)^{1/3}$, the crystal will remain magnetized even after we reduce the external magnetic field to zero. This is because when $T < \left(\frac{n(v\hbar)^3 \log 2}{12 k_B^3 I}\right)^{1/3}$, the entropy density of the phonon gas is less than $nk_B \log 2$. After we reduce the external magnetic field to zero, the entropy density of the nuclear spins only changes by an amount less than $nk_B \log 2$. This means that the nuclear spins will remain partially polarized after the external magnetic field is reduced to zero. If the nuclear spins were totally unpolarized, the entropy change in the nuclear spins would reach $nk_B \log 2$.

Statistical Mechanics 2: Quasi-particle excitations in a fermionic superfluid

A spin-1/2 Fermi gas with attractive interactions between spin up and spin down fermions forms a superfluid of fermion pairs at low temperatures. At zero temperature, all fermions are paired up, with spin up fermions pairing with spin down fermions. However, at small but non-zero temperatures some of these pairs break up, and unpaired, fermionic quasi-particle excitations exist in the system. We will here calculate their contribution to the thermodynamics of the superfluid.

At low temperatures, much smaller than the pairing gap, we can approximate the energy $\epsilon(k)$ of the spin-1/2 quasi-particle excitations at momentum $\hbar k$ by

$$\epsilon(k) \approx \Delta + \frac{\hbar^2}{2m_0}(k - k_F)^2 \tag{1}$$

where Δ is the pairing gap, $m_0 = m\frac{\Delta}{2E_F}$ the quasi-particle mass that differs from the bare mass m of free fermions, $E_F = \frac{\hbar^2 k_F^2}{2m}$ is the Fermi energy, $k_F = (3\pi^2 n)^{1/3}$ is the Fermi wavevector, $n = N/V$ is the total density of the spin-1/2 Fermi gas, N the total number and V the volume of a cubic box confining the gas. We will assume throughout this problem that $\Delta \ll E_F$, well fulfilled for conventional superconductors or weakly interacting Fermi gases, we will assume low temperatures $k_B T \ll \Delta$ and we will neglect interactions between quasi-particles.

(a) (*5 pts*) Using this approximation for $\epsilon(k)$, find the contribution from these fermionic quasi-particle excitations to the free energy of the gas

$$F_{\text{qp}}(N, V, T) \equiv -k_B T \ln Z_{\text{qp}},$$

where Z_{qp} is the partition function for quasi-particle excitations

$$Z_{\text{qp}} \equiv \sum_{\text{all qp states}} e^{-E_{\text{qp}}/k_B T}. \tag{2}$$

The sum is over all states involving any number of quasi-particle excitations, and E_{qp} is the total energy of each such state.

Write your result in the form

$$F_{\text{qp}}(N, V, T) = aNE_F \left(\frac{k_B T}{E_F}\right)^{\alpha} \left(\frac{\Delta}{E_F}\right)^{\beta} f\left(\frac{k_B T}{\Delta}\right) \tag{3}$$

where you need to find the dimensionless constant a, the exponents α and β, and the function $f(x)$ that describes the dependence of F_{qp} on the ratio of temperature $k_B T$ to superfluid gap Δ that cannot be written as a power law.

Even if you do not succeed in finding a, α, β and the functional form $f(x)$, you can do all subsequent parts by expressing your answers to each of them in terms of these constants and the function f.

Problem continued on next page.

You may find the following integral useful:

$$\int_0^\infty \mathrm{d}x\; x^2\, e^{-(x-x_0)^2} \approx \sqrt{\pi}x_0^2, \tag{4}$$

for $x_0 \gg 1$.

(b) (*2 pts*) Find the contribution to the entropy $S(N,V,T)$ and to the total energy $E_{\rm tot}(N,V,T)$ of the Fermi gas due to the quasi-particle excitations. *If you did not find the expression for $f(x)$ above, you may assume $xf'(x) \gg f(x)$ for $x \ll 1$.*

(c) (*3 pts*) In atomic Fermi gases, the superfluid gap Δ can be freely adjusted, independently of N, V and T. Let's assume the gas is initially at temperature T_0. If the gap is *adiabatically* changed from an initial value Δ_0 to a final value Δ_1, at constant N and V, what is the final temperature T_1? Assume throughout the evolution that $k_{\rm B}T \ll \Delta$. *If you did not find the expression for $f(x)$ above, you may assume $xf'(x) \gg f(x)$ and $xf''(x) \gg f'(x)$ for $x \ll 1$.*

SOLUTION:

(a) For each spin orientation σ, the possible states have $n_{i,\sigma} = 0$ or 1 quasi-particle in each of the momentum states, labeled $\vec{\boldsymbol{k}}_i$, so

$$\begin{aligned} Z_{\rm qp} &= \sum_{\{n_{i,\sigma}\}} e^{-\sum n_{i,\sigma}\epsilon(|\vec{\boldsymbol{k}}_i|)/k_{\rm B}T} \\ &= \sum_{\{n_{i,\sigma}\}} \prod_{i,\sigma} e^{-n_{i,\sigma}\epsilon(|\vec{\boldsymbol{k}}_i|)/k_{\rm B}T} \\ &= \prod_i \left(1 + e^{-\epsilon(|\vec{\boldsymbol{k}}_i|)/k_{\rm B}T}\right)^2 \\ &= \prod_{k_x}\prod_{k_y}\prod_{k_z} \left(1 + e^{-\epsilon(|\vec{\boldsymbol{k}}|)/k_{\rm B}T}\right)^2 . \end{aligned} \tag{5}$$

Then

$$F_{\rm qp} = -k_{\rm B}T \ln Z_{\rm qp} = -2k_{\rm B}T \sum_{\vec{\boldsymbol{k}}} \ln\left(1 + e^{-\epsilon(|\vec{k}|)/k_{\rm B}T}\right) \tag{6}$$

where the factor of 2 comes from the two spin components. Since $\epsilon(|\vec{k}|) > \Delta$ and $k_{\rm B}T/\Delta \ll 1$, we may expand

$$\ln\left(1 + e^{-\epsilon(|\vec{k}|)/k_{\rm B}T}\right) \approx e^{-\epsilon(|\vec{k}|)/k_{\rm B}T} \tag{7}$$

$$\begin{aligned} F_{\rm qp} &= -2k_{\rm B}T \frac{V}{2\pi^2} \int_0^\infty {\rm d}k\, k^2\, e^{-\epsilon(|\vec{k}|)/k_{\rm B}T} \\ &= -\frac{1}{\pi^2} k_{\rm B}T\, V\, e^{-\Delta/k_{\rm B}T} \int_0^\infty {\rm d}k\, k^2\, e^{-\hbar^2 \frac{(k-k_F)^2}{2m_0 k_{\rm B}T}} \\ &= -\frac{1}{\pi^2} k_{\rm B}T\, V\, e^{-\Delta/k_{\rm B}T} \left(\frac{2m_0 k_{\rm B}T}{\hbar^2}\right)^{3/2} \int_0^\infty {\rm d}x\, x^2\, e^{-(x-x_0)^2} \end{aligned} \tag{8}$$

with $x_0^2 = \frac{\hbar^2 k_F^2}{2m_0 k_{\rm B}T} = \frac{E_F}{k_{\rm B}T}\frac{2E_F}{\Delta} \gg 1$. The integral gives $\sqrt{\pi}x_0^2$ and thus

$$\begin{aligned} F_{\rm qp} &= -\frac{2}{\pi^{3/2}} E_F\, V \left(\frac{2m_0 k_{\rm B}T}{\hbar^2}\right)^{3/2} \frac{E_F}{\Delta}\, e^{-\Delta/k_{\rm B}T} \\ &= -\frac{2}{\pi^{3/2}} E_F \left(\frac{k_{\rm B}T}{E_F}\right)^{3/2} V \left(\frac{2m_0 E_F}{\hbar^2}\right)^{3/2} \frac{E_F}{\Delta}\, e^{-\Delta/k_{\rm B}T} \\ &= -\frac{2}{\pi^{3/2}} E_F \left(\frac{k_{\rm B}T}{E_F}\right)^{3/2} V k_F^3 \left(\frac{m_0}{m}\right)^{3/2} \frac{E_F}{\Delta}\, e^{-\Delta/k_{\rm B}T} \\ &= -6\sqrt{\pi} N E_F \left(\frac{k_{\rm B}T}{E_F}\right)^{3/2} \left(\frac{\Delta}{2E_F}\right)^{3/2} \frac{E_F}{\Delta}\, e^{-\Delta/k_{\rm B}T} \\ &= \boxed{-\frac{3\sqrt{2\pi}}{2} N E_F \left(\frac{k_{\rm B}T}{E_F}\right)^{3/2} \sqrt{\frac{\Delta}{E_F}}\, e^{-\Delta/k_{\rm B}T}} \end{aligned} \tag{9}$$

Thus $\boxed{a = -\frac{3\sqrt{2\pi}}{2},\ \alpha = \frac{3}{2},\ \beta = \frac{1}{2},\ f(x) = e^{-1/x}.}$

(b)

$$\begin{aligned} S &= -\left.\frac{\partial F}{\partial T}\right|_{N,V} \\ &= -\frac{3}{2}\frac{F}{T} - F\frac{\Delta}{k_{\mathrm{B}}T^2} \\ &\approx \boxed{\frac{3\sqrt{2\pi}}{2} N k_{\mathrm{B}} \left(\frac{k_{\mathrm{B}}T}{E_F}\right)^{-1/2} \left(\frac{\Delta}{E_F}\right)^{3/2} e^{-\Delta/k_{\mathrm{B}}T}} \end{aligned} \tag{10}$$

where we used $k_{\mathrm{B}}T \ll \Delta$. In terms of a, α, β, $f(x)$:

$$\begin{aligned} S &= -\left.\frac{\partial F}{\partial T}\right|_{N,V} \\ &= -\alpha\frac{F}{T} - k_{\mathrm{B}}\frac{F}{\Delta}\frac{f'(k_{\mathrm{B}}T/\Delta)}{f(k_{\mathrm{B}}T/\Delta)} \\ &\approx -k_{\mathrm{B}}\frac{F}{\Delta}\frac{f'(k_{\mathrm{B}}T/\Delta)}{f(k_{\mathrm{B}}T/\Delta)} \\ &= \boxed{-aNk_B \left(\frac{k_{\mathrm{B}}T}{E_F}\right)^{\alpha} \left(\frac{\Delta}{E_F}\right)^{\beta-1} f'\left(\frac{k_{\mathrm{B}}T}{\Delta}\right)} \end{aligned} \tag{11}$$

where we used $f'(k_{\mathrm{B}}T/\Delta) \gg \frac{\Delta}{k_{\mathrm{B}}T} f(k_{\mathrm{B}}T/\Delta)$.

We have

$$\begin{aligned} E_{\mathrm{qp}} &= F + TS = F - \frac{3}{2}F - F\frac{\Delta}{k_{\mathrm{B}}T} \\ &\approx -F\frac{\Delta}{k_{\mathrm{B}}T} \\ &= \boxed{\frac{3\sqrt{2\pi}}{2} N E_F \left(\frac{k_{\mathrm{B}}T}{E_F}\right)^{1/2} \left(\frac{\Delta}{E_F}\right)^{3/2} e^{-\Delta/k_{\mathrm{B}}T}} \end{aligned} \tag{12}$$

or in terms of a, α, β, $f(x)$:

$$\begin{aligned}
E_{\mathrm{qp}} &= F + TS = (1-\alpha)F - F\frac{k_{\mathrm{B}}T}{\Delta}\frac{f'(k_{\mathrm{B}}T/\Delta)}{f(k_{\mathrm{B}}T/\Delta)} \\
&\approx -F\frac{k_{\mathrm{B}}T}{\Delta}\frac{f'(k_{\mathrm{B}}T/\Delta)}{f(k_{\mathrm{B}}T/\Delta)} \\
&= \boxed{-aNE_F\left(\frac{k_{\mathrm{B}}T}{E_F}\right)^{\alpha+1}\left(\frac{\Delta}{E_F}\right)^{\beta-1} f'\left(\frac{k_{\mathrm{B}}T}{\Delta}\right)}
\end{aligned} \tag{13}$$

(c) Since the change occurs adiabatically, the entropy is constant and thus

$$T^{-1/2}\Delta^{3/2}e^{-\Delta/k_{\mathrm{B}}T} = \text{const.} \tag{14}$$

The change in the exponential will dominate, yielding $\frac{\Delta}{k_{\mathrm{B}}T} = \text{const.}$ The final temperature will thus be $\boxed{T_1 = T_0\frac{\Delta_1}{\Delta_0}.}$

The more formal derivation could employ the Maxwell relation:

$$\begin{aligned}
\left.\frac{\partial k_{\mathrm{B}}T}{\partial \Delta}\right|_{N,V,S} &= -\frac{\left.\frac{\partial S}{\partial \Delta}\right|_{N,V,T}}{\left.\frac{\partial S}{\partial k_{\mathrm{B}}T}\right|_{N,V,\Delta}} \\
&= -\frac{\frac{3}{2}\left(\frac{\Delta}{k_{\mathrm{B}}T}\right)^{1/2}e^{-\Delta/k_{\mathrm{B}}T} - \left(\frac{\Delta}{k_{\mathrm{B}}T}\right)^{3/2}e^{-\Delta/k_{\mathrm{B}}T}}{-\frac{1}{2}\left(\frac{\Delta}{k_{\mathrm{B}}T}\right)^{3/2}e^{-\Delta/k_{\mathrm{B}}T} + \left(\frac{\Delta}{k_{\mathrm{B}}T}\right)^{5/2}e^{-\Delta/k_{\mathrm{B}}T}} \\
&\approx \frac{k_{\mathrm{B}}T}{\Delta}
\end{aligned} \tag{15}$$

and thus

$$\begin{aligned}
\frac{\mathrm{d}T}{T} &= \frac{\mathrm{d}\Delta}{\Delta} \\
\log\left(\frac{T_1}{T_0}\right) &= \log\left(\frac{\Delta_1}{\Delta_0}\right)
\end{aligned}$$

and again $\boxed{T_1 = T_0\frac{\Delta_1}{\Delta_0}.}$

In terms of a, α, β and $f(x)$: The short solution is to write down $S = \text{const.}$:

$$T^{\alpha}\Delta^{\beta-1}f'\left(\frac{k_{\mathrm{B}}T}{\Delta}\right) = \text{const.} \tag{16}$$

and to take from the hint that the change in $f'(x)$ will dominate the change in the other terms, so that we again require $\frac{k_{\mathrm{B}}T}{\Delta} = \text{const.}$ and obtain $\boxed{T_1 = T_0\frac{\Delta_1}{\Delta_0}}$.

More formally:

$$\begin{aligned}
\left.\frac{\partial k_{\mathrm{B}}T}{\partial \Delta}\right|_{N,V,S} &= -\frac{\left.\frac{\partial S}{\partial \Delta}\right|_{N,V,T}}{\left.\frac{\partial S}{\partial k_{\mathrm{B}}T}\right|_{N,V,\Delta}} \\
\left.\frac{\partial S}{\partial \Delta}\right|_{N,V,T} &= -(\beta-1)aNk_B\left(\frac{k_{\mathrm{B}}T}{E_F}\right)^{\alpha}\frac{\Delta^{\beta-2}}{E_F^{\beta-1}}f'\left(\frac{k_{\mathrm{B}}T}{\Delta}\right) \\
&\quad +aNk_B\left(\frac{k_{\mathrm{B}}T}{E_F}\right)^{\alpha}\left(\frac{\Delta}{E_F}\right)^{\beta-1}f''\left(\frac{k_{\mathrm{B}}T}{\Delta}\right)\frac{k_{\mathrm{B}}T}{\Delta^2} \\
\left.\frac{\partial S}{\partial k_{\mathrm{B}}T}\right|_{N,V,\Delta} &= -\alpha aNk_B\frac{(k_{\mathrm{B}}T)^{\alpha-1}}{E_F^{\alpha}}\left(\frac{\Delta}{E_F}\right)^{\beta-1}f'\left(\frac{k_{\mathrm{B}}T}{\Delta}\right) \\
&\quad -aNk_B\left(\frac{k_{\mathrm{B}}T}{E_F}\right)^{\alpha}\left(\frac{\Delta}{E_F}\right)^{\beta-1}f''\left(\frac{k_{\mathrm{B}}T}{\Delta}\right)\frac{1}{\Delta}
\end{aligned} \tag{17}$$

Since $f''(k_{\mathrm{B}}T/\Delta) \gg \frac{\Delta}{k_{\mathrm{B}}T}f'(k_{\mathrm{B}}T/\Delta)$, we only keep the terms involving f''. Thus

$$\left.\frac{\partial k_{\mathrm{B}}T}{\partial \Delta}\right|_{N,V,S} = \frac{f''\left(\frac{k_{\mathrm{B}}T}{\Delta}\right)\frac{k_{\mathrm{B}}T}{\Delta^2}}{f''\left(\frac{k_{\mathrm{B}}T}{\Delta}\right)\frac{1}{\Delta}} = \frac{k_{\mathrm{B}}T}{\Delta} \tag{18}$$

and the result follows: $\boxed{T_1 = T_0\frac{\Delta_1}{\Delta_0}.}$

SECTION IV: QUANTUM MECHANICS

Quantum Mechanics 1: A Quantum Spin Chain

Consider a one-dimensional chain of N spin-1/2 particles coupled through the Hamiltonian

$$H = J \sum_{i=1}^{N-1} \vec{S}_i \cdot \vec{S}_{i+1}, \tag{1}$$

where $\vec{S} = (S_x, S_y, S_z)$ are the usual spin operators for a spin-1/2 particle, $J > 0$ is a **positive** constant, and $N \gg 1$.

In a famous 1931 paper, Hans Bethe showed that for this Hamiltonian, the ground state energy per particle $E_{\rm GS}/N \equiv E_0$ was equal to $-\hbar^2 J(\log 2 - 1/4) \approx -0.433\hbar^2 J$. You will not be required to reproduce this result. Instead, you will determine upper and lower bounds for the ground state energy per particle.

(a) (*2 pts*) If the spin operators are treated as classical spin vectors with $|\vec{S}| = \hbar/2$, what is the ground state spin configuration and what is the ground state energy per particle E_0?

(b) (*5 pts*) Consider the trial wave function

$$|\Psi\rangle = \bigotimes_{i=\text{odd}} |i, i+1\rangle_0 = |1,2\rangle_0 \otimes |3,4\rangle_0 \otimes \cdots \otimes |N-1, N\rangle_0\,, \tag{2}$$

where $|i,j\rangle_0$ is the spin singlet state formed from the spins on sites i and j. Use this state to find an **upper** bound on E_0.

(c) (*3 pts*) Prove the following **lower** bound on E_0:

$$-\frac{3}{4}\hbar^2 J \le E_0. \tag{3}$$

•

(a) (*2 pts*) Because $J > 0$, the interaction is most attractive between opposing spins. Thus, the state of minimum energy is $|\uparrow\downarrow\uparrow\downarrow\cdots\rangle$, where $\uparrow$ and $\downarrow$ refer to an arbitrary direction in space. For opposing c-number spins of magnitude $\hbar/2$, $\vec{S}_i \cdot \vec{S}_{i+1} = -\hbar^2/4$, so in the large N limit,

$$\boxed{E_0 = -\frac{1}{4}\hbar^2 J.} \tag{4}$$

(b) (*5 pts*) We can obtain a bound from the variational theorem which states that the ground state energy is less than $\langle\Psi| H |\Psi\rangle$ for any normalized trial state $|\Psi\rangle$.

Labeling the positions by 1,2, ..., the suggested trial state is

$$|\Psi\rangle = |12_0\rangle\,|34_0\rangle\,|56_0\rangle\cdots, \tag{5}$$

where the pairwise singlet state is defined as $|ij_0\rangle = \frac{1}{\sqrt{2}}\left(|i_\uparrow j_\downarrow\rangle - |i_\downarrow j_\uparrow\rangle\right)$.

The task is to calculate the expectation value of H in the state $|\Psi\rangle$. Two types of terms occur in the sum over i in H: ones in which the coupled spins in H appear in the same singlet, for example,

$$E_{1\cdot 2} = J\,\langle\Psi|\,\vec{S}_1 \cdot \vec{S}_2\,|\Psi\rangle\,, \tag{6}$$

and ones in which the coupled spins in H appear in two different singlets, for example,

$$E_{2\cdot 3} = J\,\langle\Psi|\,\vec{S}_2 \cdot \vec{S}_3\,|\Psi\rangle\,. \tag{7}$$

The first type of term is simple to evaluate,

$$E_{1\cdot 2} = J\,\langle 12_0|\,\vec{S}_1 \cdot \vec{S}_2\,|12_0\rangle = -\frac{3}{4}\hbar^2 J, \tag{8}$$

where we have used the fact that $\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left((\vec{S}_1 + \vec{S}_2)^2 - \vec{S}_1^2 - \vec{S}_2^2\right)$, the singlet state has total spin 0, and $\vec{S}^2 = -(3/4)\hbar^2$ for a spin-1/2 particle.

The second type of term can be simplified by realizing that the operators only act on the relevant spin states

$$E_{2\cdot 3} = J\,\langle 12_0\,34_0|\,\vec{S}_2 \cdot \vec{S}_3\,|12_0 34_0\rangle = J\,\langle 12_0|\,\vec{S}_2\,|12_0\rangle \cdot \langle 34_0|\,\vec{S}_3\,|34_0\rangle\,. \tag{9}$$

Because the singlet states are rotationally symmetric, $\langle 12_0|\,\vec{S}_2\,|12_0\rangle = 0$. More explicitly,

$$S_{2z}\,|12_0\rangle = S_{2z}\frac{1}{\sqrt{2}}\left(|1_\uparrow 2_\downarrow\rangle - |1_\downarrow 2_\uparrow\rangle\right) = \frac{1}{\sqrt{2}}\left(-\,|1_\uparrow 2_\downarrow\rangle - |1_\downarrow 2_\uparrow\rangle\right), \tag{10}$$

which is orthogonal to $|12_0\rangle$, and similarly for S_{2x} and S_{2y}. Thus, terms of the second type contribute zero to the sum.

As a result, each singlet pair contributes $-\frac{3}{4}\hbar^2 J$ to $\langle\Psi| H |\Psi\rangle$. There are $N/2$ pairs, so we obtain the upper bound

$$\boxed{E_0 \leq -\frac{3}{8}\hbar^2 J.} \tag{11}$$

(c) *(3 pts)* Consider the expectation value of any pair $\vec{S}_i \cdot \vec{S}_{i+1}$ in any state $|\alpha\rangle$. The expectation value of an operator cannot be less than its lowest eigenvalue λ_{min}:

$$\langle\alpha| \vec{S}_i \cdot \vec{S}_{i+1} |\alpha\rangle \geq \lambda_{\text{min}}. \tag{12}$$

The eigenvalues of $\vec{S}_i \cdot \vec{S}_{i+1}$ are

$$\frac{1}{2}\hbar^2 \left(s(s+1) - \frac{3}{4} - \frac{3}{4} \right), \tag{13}$$

which yields $-\frac{3}{4}\hbar^2$ (for the $s = 0$ singlet state) and $+\frac{1}{4}\hbar^2$ (for the $s = 1$ triplet state), so $\langle\alpha| \vec{S}_i \cdot \vec{S}_{i+1} |\alpha\rangle \geq -\frac{3}{4}\hbar^2$. This applies to all N terms in H, so we obtain the lower bound

$$\boxed{E_0 \geq -\frac{3}{4}\hbar^2 J.} \tag{14}$$

Quantum Mechanics 2: Anomalous Magnetic Moment of the Electron

The gyromagnetic factor of the electron g determines the relationship between the electron magnetic moment $\vec{\mu}$ and the electron spin $\vec{S}$,

$$\vec{\mu} = g\frac{e}{2m}\vec{S}, \tag{1}$$

where e is the electron charge and m is the electron mass. Famously, the Dirac equation predicts $g = 2$, but in quantum electrodynamics, the electron picks up an anomalous magnetic moment $g = 2(1 + a)$, where the current experimental value is $a = 0.00115965218076(27)$.

One way to experimentally measure a is to allow a beam of electrons to interact with a constant magnetic field $\vec{B} = B\hat{z}$ via the Hamiltonian

$$H = \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 - \vec{\mu}\cdot\vec{B}, \tag{2}$$

where $\vec{A}$ is the vector potential. The electrons are confined to the x–y plane, and you can ignore any electron–electron interactions. The electrons will exhibit cyclotron motion with frequency $\omega = eB/m$, but they will also exhibit spin precession with a slightly different frequency. In this problem, you will show how to use this phenomenon to extract a.

(a) (*2 pts*) Verify the commutation relations

$$[v_x, H] = i\hbar\omega v_y, \qquad [v_y, H] = -i\hbar\omega v_x, \tag{3}$$

where $\vec{v} = (\vec{p} - e\vec{A})/m$ is the gauge-invariant velocity operator. [*Hint*: Because $\vec{v}$ is gauge invariant, you are free to choose any gauge for $\vec{A}$ you wish.]

(b) (*6 pts*) Consider the two expectation values

$$C_1(t) = \langle S_x v_x + S_y v_y\rangle, \qquad C_2(t) = \langle S_x v_y - S_y v_x\rangle. \tag{4}$$

Derive a set of coupled differential equations that describe the time evolution of $C_1(t)$ and $C_2(t)$. In the special case that $a = 0$ (i.e. $g = 2$), verify that $C_1(t)$ and $C_2(t)$ do not change with time.

(c) (*2 pts*) A beam of electrons of velocity $\vec{v}$ is prepared at time $t = 0$ in a spin state with known values of $C_1(0)$ and $C_2(0)$. The beam interacts with a magnetic field $\vec{B} = B\hat{z}$ between $t = 0$ and $t = T$. The expectation value $C_1(T)$ is experimentally measured to be periodic with period $2\pi/\Omega$ (i.e. $C_1(T) = C_1(T + 2\pi/\Omega)$). Use this information to determine the value of a in terms of Ω and other physical parameters.

•

(a) (*2 pts*) Following the hint, it is easiest to choose a concrete gauge, and we will select Landau gauge with $\vec{A} = (0, Bx, 0)$. In this gauge, the velocity operators are

$$v_x = \frac{p_x}{m}, \qquad v_y = \frac{1}{m}(p_y - eBx), \tag{5}$$

and they have a commutation relation

$$[v_x, v_y] = \frac{-eB}{m^2}[p_x, x] = \frac{i\hbar eB}{m^2} = i\hbar\omega/m, \qquad [v_x, v_x] = [v_y, v_y] = 0. \tag{6}$$

The Hamiltonian is

$$H = \frac{m}{2}(v_x^2 + v_y^2) - (1+a)\frac{e}{m}BS_z. \tag{7}$$

Using the fact that

$$\frac{1}{2}[v_x, v_y^2] = \frac{1}{2}\left(v_y[v_x, v_y] + [v_x, v_y]v_y\right) = i\hbar\omega v_y/m, \tag{8}$$

the desired commutation relations are

$$[v_x, H] = \frac{m}{2}[v_x, v_y^2] = \boxed{i\hbar\omega v_y,} \qquad [v_y, H] = \frac{m}{2}[v_y, v_x^2] = \boxed{-i\hbar\omega v_x.} \tag{9}$$

(b) (*6 pts*) The Ehrenfest theorem tells us that

$$i\hbar\frac{d}{dt}\langle\mathcal{O}\rangle = \langle[\mathcal{O}, H]\rangle, \tag{10}$$

for any Hermitian operator $\mathcal{O}$. So we need to calculate a number of commutation relations. For arbitrary a, note that

$$[S_x, H] = -(1+a)\omega[S_x, S_z] = +i(1+a)\hbar\omega S_y, \qquad [S_y, H] = -i(1+a)\hbar\omega S_x. \tag{11}$$

Thus, we have

$$\begin{aligned}
[S_x v_x, H] &= S_x[v_x, H] + [S_x, H]v_x = i\hbar\omega(S_x v_y + (1+a)S_y v_x) && (12)\\
[S_x v_y, H] &= S_x[v_y, H] + [S_x, H]v_y = i\hbar\omega(-S_x v_x + (1+a)S_y v_y) && (13)\\
[S_y v_x, H] &= S_y[v_x, H] + [S_y, H]v_x = i\hbar\omega(S_y v_y - (1+a)S_x v_x) && (14)\\
[S_y v_y, H] &= S_y[v_y, H] + [S_y, H]v_y = i\hbar\omega(-S_y v_x - (1+a)S_x v_y) && (15)
\end{aligned}$$

For convenience, we define

$$\mathcal{O}_1 = S_x v_x + S_y v_y, \qquad \mathcal{O}_2 = S_x v_y - S_y v_x, \tag{16}$$

and we see that

$$[\mathcal{O}_1, H] = -ia\hbar\omega\mathcal{O}_2, \qquad [\mathcal{O}_2, H] = +ia\hbar\omega\mathcal{O}_1. \tag{17}$$

Thus, the time evolution of $C_1(t)$ and $C_2(t)$ are dictated by

$$\boxed{\frac{d}{dt}C_1(t) = -a\omega C_2(t), \qquad \frac{d}{dt}C_2(t) = a\omega C_1(t).} \tag{18}$$

For $a = 0$, $\frac{d}{dt}C_i(t) = 0$, so implies that $C_1(t)$ and $C_2(t)$ are constant in time.

(c) (*2 pts*) We can rewrite Eq. (18) as a second order differential equation for $C_1(t)$

$$\frac{d^2}{dt^2}C_1(t) = -(a\omega)^2 C_1(t). \tag{19}$$

A general solution to this equation is

$$C_1(t) = A\cos(a\omega t + \varphi), \tag{20}$$

where A and φ are integration constants. This function has period $2\pi/a\omega$, so if the measured period is $2\pi/\Omega$, we can extract a as

$$\boxed{a = \frac{\Omega}{\omega}.} \tag{21}$$