MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS

Academic Programs
Room 4-315

Phone: (617) 253-4851
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DOCTORAL GENERAL EXAMINATION

WRITTEN EXAM

September 4, 2015

DURATION: 75 MINUTES PER SECTION

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DOCTORAL GENERAL EXAMINATION

WRITTEN EXAM - CLASSICAL MECHANICS

September 4, 2015

DURATION: 75 MINUTES

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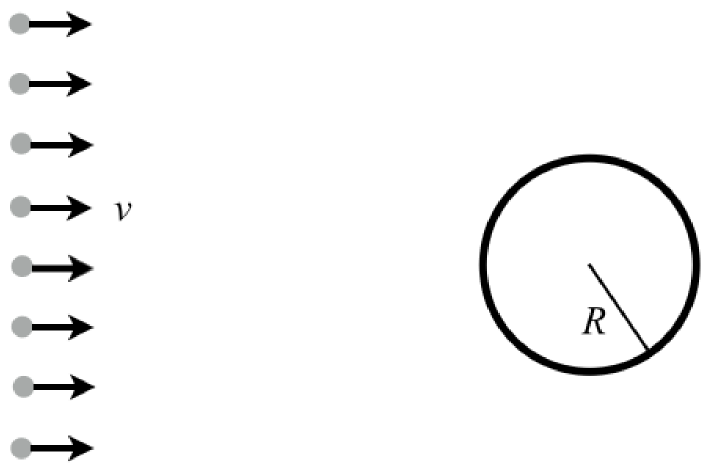
2. If you decide at this time not to take this section of the exam, please inform the faculty proctor. ONCE YOU BEGIN THE EXAM, IT WILL BE COUNTED.
3. Calculators may not be used.
4. No books or reference materials may be used.

Classical Mechanics 1: Striking the sphere

A hard sphere of radius R is fixed at the center of a spherically symmetric potential $U(r)$. The potential $U(r)$ declines monotonically to zero as $r \longrightarrow \infty$.

A beam of test particles is aimed at the sphere from a great distance. All the particles are moving parallel to one another with speed $v \ll c$. The number density of particles is n particles/cm^3, and each particle has a mass m.



(a) (*4 pts*) Calculate the cross-section for striking the sphere, in terms of $U(R)$, v, and m.

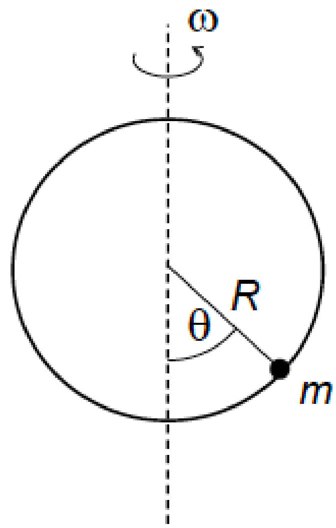
(b) (*2 pts*) Is it possible for the cross-section to vanish? If so, under what conditions?

(c) (*4 pts*) Now take $U(r) = -GMm/r$, the gravitational potential between the sphere of mass M and the test particle of mass m.

Whenever a particle strikes the sphere, the particle sticks to the sphere and increases the sphere's mass but without appreciably changing the radius. Calculate the time required for the sphere's mass to increase from an initial value of M_i to a final value of M_f.

Classical Mechanics 2: Particle sliding on a rotating circular wire

A pointlike particle of mass m is constrained to move on a circular wire of radius R. The particle can slide without friction. The circular wire spins with constant angular speed ω about the vertical diameter. The force of gravity mg is acting on the particle.



(a) (*2 pts*) Write the Lagrangian for the system using θ as the generalized coordinate. Identify an effective potential $V(\theta)$.

(b) (*2 pts*) Write down the Euler Lagrange equation and derive the equation of motion in terms of m, g, R, ω, and θ. (Do not solve it.)

(c) (*2 pts*) Find constant values $\theta_i, i = 1, 2, ...,$ for which $\theta(t) = \theta_i$ is a stationary solution of the equation of motion. Express your answer in terms of ω, R, and g. Do all solutions exist for all values of ω?

(d) (*4 pts*) Consider now small displacements from each of the equilibrium values θ_i identified in part (c). Determine, as a function of ω whether such displacements lead to stable or unstable oscillations. If the small-amplitude oscillation is stable, determine the corresponding oscillation frequency Ω_i. Summarize your results in a graph where you plot the θ_i as functions of ω and label the various parts of the curves as "stable" or "unstable".

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DOCTORAL GENERAL EXAMINATION

WRITTEN EXAM - ELECTRICITY AND MAGNETISM

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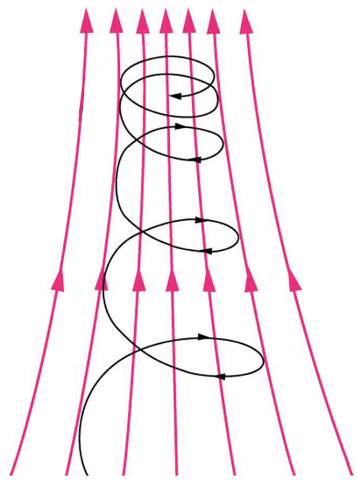
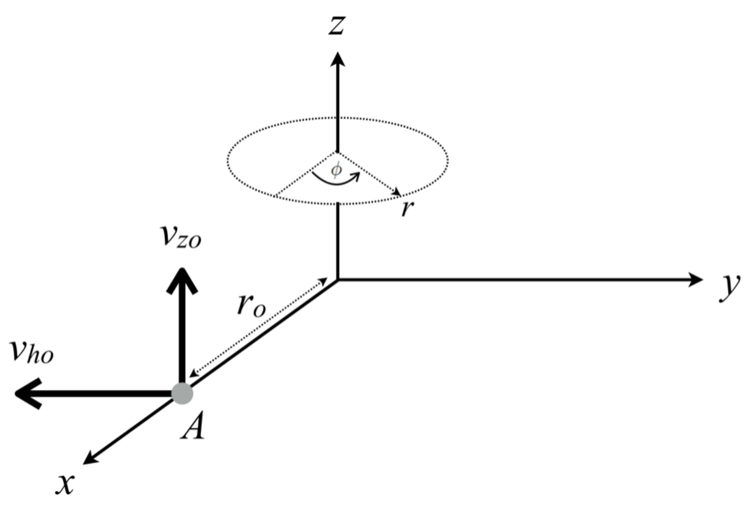
Electromagnetism 1: Magnetic mirror

When a charged particle orbits around magnetic field lines while also drifting along the field into a region of higher field strength, then the particle experiences a force that reduces the component of velocity parallel to the field. This force slows the motion along the field lines and may reverse it. This is the basis of a "magnetic mirror," illustrated below.



In this problem we will investigate this phenomenon using a cylindrical coordinate system in which the z-axis is the symmetry axis, r denotes the cylindrical radius (the distance from the z-axis), and ϕ is the azimuthal angle measured from the x-axis, as illustrated below.



You may find it useful to remember

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}. \tag{1}$$

Consider a magnetic field $\vec{B}$ which is axially symmetric around the z-axis. The z-component is

$$B_z(r, z) = B_0 + B_0' z, \tag{2}$$

where B_0' is a constant.

We inject a particle with mass m and charge $q > 0$ at point A, located at $x = r_0$ and $y = z = 0$. The particle's initial velocity has both horizontal and vertical components:

$$\vec{v} = -v_{\text{h}0}\hat{y} + v_{\text{z}0}\hat{z}, \tag{3}$$

where $v_{\text{h}0} = \frac{q}{mc} B_0 r_0$ and $v_{\text{z}0} \ll v_{\text{h}0}$. Both components of the velocity are non-relativistic.

(a) (*1 pt*) Calculate the radial component of the magnetic field, $B_r(r, z)$.

(b) (*1 pt*) Show that throughout the subsequent motion,

$$v^2(t) = v_{h0}^2 + v_{z0}^2. \tag{4}$$

Now assume that during each orbit around the z-axis, the horizontal velocity is nearly in the $-\hat{\phi}$ direction and the change in radius Δr of the orbit is negligible compared to the instantaneous radius r.

(c) (*1 pt*) What is the radius r of the orbit as a function of v_ϕ, B_z and physical constants?

(d) (*2 pts*) Find an equation for the z-dependence of v_h (the horizontal speed) as the particle drifts in the z-direction. One way to do so is to write the equation of motion for the $\hat{\phi}$ component of the velocity, and then use $v_z dt = dz$.

(e) (*3 pts*) Using your result from part (c), integrate your equation from part (d) to show that the horizontal speed varies with z as

$$\frac{v_h(z)}{v_{h0}} = \sqrt{\frac{B_z(z)}{B_0}}. \tag{5}$$

(f) (*2 pts*) Find the value of z for which $v_z = 0$. This is the reflection point of the magnetic mirror, where the spiraling particle stops its upward motion and starts moving downward.

Express your answer entirely in terms of B_0, B_0', v_{h0} and v_{z0}.

Electromagnetism 2: Electromagnetic waves in a plasma

A plane electromagnetic wave of angular frequency ω propagates in a uniform plasma with electron density N_e. The plasma is locally neutral, $\rho = 0$. The electromagnetic wave generates periodic currents within the plasma that, as the problem will show, modify the index of refraction of the medium compared to that of the vacuum.

Assume the plasma has no resistivity and neglect radiation pressure effects as well as currents due to the ions.

(a) (*3 pts*) Relate the current $\vec{J}(\vec{r}, t)$ in the plasma to the wave's electric field $\vec{E}(\vec{r}, t)$ or derivatives thereof. Assume magnetic forces can be neglected.

(b) (*3 pts*) Write down the appropriate Maxwell equations and derive the wave equation. Find the dispersion relation $\omega(k)$ and the lowest frequency electromagnetic wave that can propagate the plasma.

(c) (*2 pts*) Find the phase and group velocity for electromagnetic waves in the plasma. Compare those velocities with c, the speed of light in vacuum.

(d) (*1 pt*) Find the index of refraction n of the plasma as a function of frequency.

(e) (*1 pt*) If a plane electromagnetic wave is incident on a plane interface between a vacuum and the plasma, what is the critical angle for total reflection, measured from the normal to the interface?

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DOCTORAL GENERAL EXAMINATION

WRITTEN EXAM - STATISTICAL MECHANICS

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Statistical Mechanics 1: Potts Model

The q-state Potts model generalizes the Ising model. There is a variable $\sigma_i \in \{1, 2, \ldots, q\}$ at each lattice site. The Hamiltonian is given by a sum over nearest neighbors:

$$H_{\text{Potts}} = -\frac{3J}{2} \sum_{\langle i,j \rangle} \delta_{\sigma_i, \sigma_j}. \tag{1}$$

There are N lattice sites. Assume $J > 0$. For parts (a) and (b), $q \geq 2$ is arbitrary, and for (c) and (d), we assume $q = 3$.

(a) (*1 pt*) What is the entropy of the system at $T = 0$?

(b) (*3 pts*) For this part only, suppose the N sites are arranged on a line with open boundary conditions. Write down the (exact) free energy.

(c) (*1 pt*) For the $q = 3$ case, show that the model is equivalent to

$$H = -J \sum_{\langle i,j \rangle} \vec{s}_i \cdot \vec{s}_j \tag{2}$$

with each $\vec{s}_i$ restricted to take values in the set

$$\vec{s}_i \in \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix} \right\} \tag{3}$$

(d) (*5 pts*) Let $\vec{m}$ denote the mean-field magnetization $\sum_i \vec{s}_i / N$. Suppose that each site in the lattice interacts with z other sites. Use a mean-field approximation to derive an expression for the free energy. (Your answer should be left in terms of a solution to a transcendental equation.) In other words, replace the true nearest-neighbor interactions with an approximation in which every site interacts with every other site with an interaction strength rescaled appropriately. Is there a first-order (i.e. discontinuous in m) phase transition?

[Hint: it may be helpful to consider $\vec{m}$ of the form $(m, 0)$.]

Statistical Mechanics 2: Magnetic susceptibility of a spinful Boltzmann gas

Consider a classical ideal gas of N spin-1/2 atoms moving in a container of volume V. In the presence of a weak external magnetic field H the energy of the nth such atom may be taken to be

$$E(\vec{p}_n, \sigma_n) = \frac{\vec{p}_n^{\,2}}{2m} - \gamma H \sigma_n \tag{1}$$

Here $\sigma_n = \pm 1$ describes the two possible spin orientations of the atom and $\vec{p}_n$ is the momentum of the atom. γ is a positive constant.

(a) (*2 pts*) Calculate the change in free energy due to the magnetic field.

(b) (*3 pts*) Calculate the average magnetization per atom $\langle M \rangle = \frac{1}{N}\gamma\langle\sum_n \sigma_n\rangle$.

(c) (*3 pts*) Calculate the variance in the magnetization $\langle M^2 \rangle - \langle M \rangle^2$.

(d) (*2 pts*) Use the above to calculate the magnetic susceptibility $\chi = \frac{d\langle M \rangle}{dH}$. How is the susceptibility related to the variance?

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WRITTEN EXAM - QUANTUM MECHANICS

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Quantum Mechanics 1: A spin-1/2 particle in a magnetic field

A spin-1/2 particle interacts with a magnetic field via the Hamiltonian

$$H = \vec{\sigma} \cdot \vec{B}, \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

(We absorb the other relevant constants into the definition of $\vec{B}$.)

(a) (*5 pts*) Find the eigenvalues and eigenvectors of H. You may find it convenient to write $\vec{B}$ as

$$\vec{B} = \begin{pmatrix} B\sin(\theta)\cos(\phi) \\ B\sin(\theta)\sin(\phi) \\ B\cos(\theta) \end{pmatrix}.$$

(b) (*5 pts*) Let $\vec{B} = B_0\hat{z}$. Suppose at time $t = 0$ the spin is initially pointing in the $+\hat{x}$ direction. After time $t = T$, the spin is measured along the $\hat{y}$ direction. What are the possible outcomes and what are their probabilities?

Quantum Mechanics 2: Two interacting fermions

Consider two identical fermions of mass m interacting with each other through an attractive harmonic potential. The Hamiltonian is

$$H = \frac{\mathbf{p}_1^2}{2m} + \frac{\mathbf{p}_2^2}{2m} + \frac{k}{2}\left(\mathbf{x}_1 - \mathbf{x}_2\right)^2 \tag{1}$$

$\mathbf{x}_{1,2}$ are the coordinates of the two fermions, and $\mathbf{p}_{1,2}$ are the conjugate momenta. For simplicity assume that the spins of both fermions are polarized in the same direction (e.g, in the $\uparrow$ direction) so that we may ignore the spin degree of freedom.

(a) (*2 pts*) State the restriction imposed by Fermi statistics on acceptable wave functions $\psi(\mathbf{x}_1, \mathbf{x}_2)$ of this system.

(b) (*2 pts*) Rewrite H in terms of center-of-mass and relative coordinates.

(c) (*3 pts*) Ignoring the restriction imposed by Fermi statistics what is the ground state energy? What is the energy of the first excited bound state?

(d) (*3 pts*) Including the effects of Fermi statistics what is the ground state energy? What is the degeneracy of the ground state?