MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS

Academic Programs
Room 4-315

Phone: (617) 253-4841
Fax: (617) 258-8319

DOCTORAL GENERAL EXAMINATION

WRITTEN EXAM

Thursday, August 26, 2021

DURATION: 75 MINUTES PER SECTION

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS

Academic Programs
Room 4-315

Phone: (617) 253-4841
Fax: (617) 258-8319

DOCTORAL GENERAL EXAMINATION

WRITTEN EXAM - CLASSICAL MECHANICS

Thursday, August 26, 2021

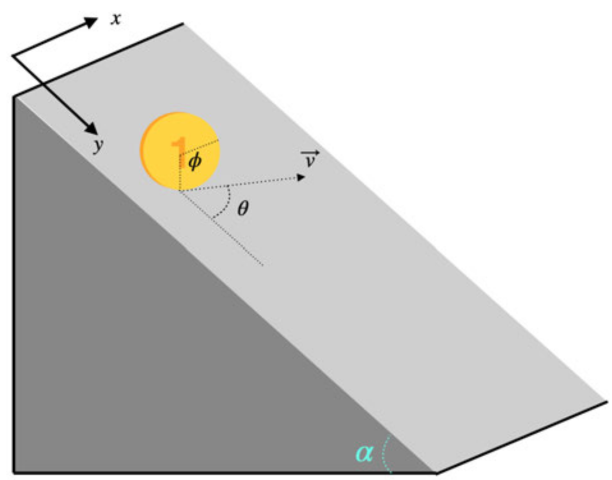
DURATION: 75 MINUTES

1. This examination has two problems. Read both problems carefully before making your choice. Submit ONLY one problem. IF YOU SUBMIT MORE THAN ONE PROBLEM FROM A SECTION, BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.
2. If you decide at this time not to take this section of the exam, please inform the faculty proctor. ONCE YOU BEGIN THE EXAM, IT WILL BE COUNTED.
3. Calculators may not be used.
4. No books or reference materials may be used.

Classical Mechanics 1: The skateboard medal

Skateboarding was one of the new disciplines introduced at the Tokyo 2020 Olympic games. The winner of the gold medal is getting ready for their mechanics exam by letting the medal roll along the skateboard ramp and analyzing its motion.

To simplify the math, they, and us, will assume that the medal is a thin disk of constant density, with total mass M and radius R (that is: we entirely neglect its thickness). We define coordinates x and y on the (flat) surface of the ramp as in the Figure.



We will use an angle ϕ to measure rotations around the medal axis and θ to measure rotations around an axis perpendicular to the ramp and passing through the contact point (see Figure). We define θ such that $\theta = 0$ if the instantaneous velocity $\vec{v}$ of the medal is along the $+y$ direction. We call α the angle that the incline makes with the horizontal and take the acceleration of gravity to be g, directed downward. We will assume that the medal is constrained such that its plane is always perpendicular to the ramp, and that it rolls without slipping.

a) (10 pts) Calculate all the non-zero compontents of the moment of inertia tensor for the medal relative to a coordinate system whose origin is at the center of the medal, with the Z axis perpendicular to the medal and the X and Y axes in the plane of the medal;

b) (10 pts) Write down two independent constraints that might relate some or all of ϕ, θ, x, y and/or their time derivatives;

c) (20 pts) Write the Lagrangian of the system in terms of some or all of ϕ, θ, x, y and/or their time derivatives;

d) (35 pts) Find the equations of motion for $\theta(t)$ and $\phi(t)$. Your answer cannot contain x or y;

e) (25 pts) Find explicit solutions for $x(t)$ and $y(t)$. Call $\omega \equiv \mathrm{d}\theta/\mathrm{dt}|_{\mathrm{t=0}} \neq 0$. You can use the initial conditions: $x(t = 0) = y(t = 0) = \phi(t = 0) = \theta(t = 0) = \mathrm{d}\phi/\mathrm{dt}|_{\mathrm{t=0}} = 0$

Potentially useful identities

$$\int_0^a \frac{\mathrm{dx}}{1+\cos(2x)} = \frac{\tan a}{2}; \int_0^a \mathrm{dx}\sqrt{\frac{1+\cos(2\mathrm{x})}{\cos(2\mathrm{x})-\cos(2\mathrm{a})}} = \frac{\pi}{2}$$

$$\int_0^a \frac{\mathrm{dx}}{\sqrt{\cos x - \cos(a)}} \simeq \frac{\pi}{\sqrt{2}}\left(1+\frac{k^2}{4}+\mathcal{O}(k^4)\right), \text{ with } \mathrm{k} \equiv \sin(\mathrm{a}/2)$$

$$\int_a^\pi \mathrm{dx}\frac{\sin(\mathrm{x}/2)}{\sqrt{\cos(\mathrm{a})-\cos(\mathrm{x})}} = \frac{\pi}{\sqrt{2}}; \int_\mathrm{a}^\pi \mathrm{dx}\,\mathrm{x}\cos(\mathrm{x}) = \frac{1}{2}\left(-\mathrm{a}^2+\pi^2-2\sin\mathrm{a}\right)$$

$$\int \mathrm{d}x\sqrt{1-x^2} = \frac{1}{2}\left[x\sqrt{1-x^2}+\arcsin x\right]; \int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x$$

$$\int \frac{x^2\mathrm{d}x}{\sqrt{1-x^2}} = \frac{1}{2}\left[\arcsin x - x\sqrt{1-x^2}\right]; \int \frac{x^4\mathrm{d}x}{\sqrt{1-x^2}} = \frac{1}{8}\left[3\arcsin x - x\sqrt{1-x^2}(2x^2+3)\right]$$

$$\int \mathrm{d}x\sqrt{2\left(1-\cos x\right)}\mathrm{d}x = -2\sqrt{2-2\cos x}\cot(x/2)$$

$$\int \mathrm{dx}\,\sin^2(x) = \frac{x}{2}-\frac{1}{4}\sin(2x)$$

$$\int \mathrm{dx}\,\cos(x)\sin(x) = -\frac{1}{2}\cos^2 x$$

$$2\cos^2 x = 1+\cos 2x$$

$$2\sin^2 x = 1-\cos 2x$$

$$\tan x = \frac{1-\cos 2x}{\sin 2x}$$

$$\sin(a\pm b) = \sin a\cos b \pm \cos a \sin b$$

$$\cos(a\pm b) = \cos a\cos b \mp \sin a\sin b$$

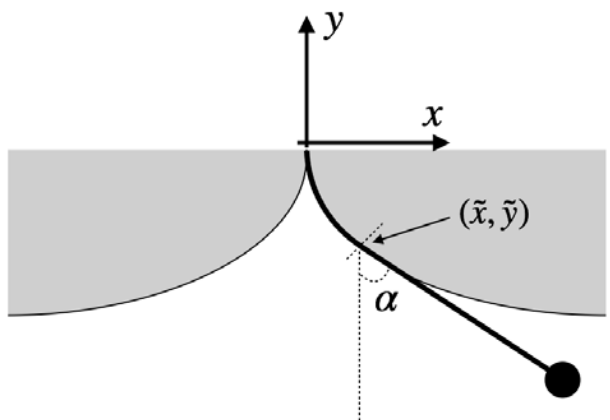
Classical Mechanics 2: Two pendula

[Throughout this problem you can assume a constant gravitational field, with acceleration $\vec{g} = -g\hat{y}$, with $g > 0$.]

Consider first a simple pendulum with mass M hanging at the end of an inextensible and massless string of length $L = 4R$. The pendulum is released at rest from an initial angle γ_0, which you may *not* consider small.

a) (25 pts) Find the period of oscillation and explicitly comment on whether it increases, stays constant, or decreases with γ_0.

Now consider a different pendulum, also of mass M hanging at the end of an inextensible and massless string of length $L = 4R$. The ceiling that the string of this second pendulum is fixed to is not flat. Instead, it has the shape shown in figure:



As the pendulum swings, the string can wrap partially around the ceiling. We call α the angle between the part of string that is not wrapped around the ceiling and the vertical direction. The symmetric bump in the ceiling is constructed such that the length of the string that is touching the ceiling – $\ell(\alpha)$ – is a simple function of α:

$$\ell(\alpha) = 4R(1 - \cos\alpha),$$

while the (x, y) coordinates of the point where the string leaves the ceiling are:

$$\tilde{x} = R(2\alpha - \sin 2\alpha) \;\; ; \;\; \tilde{y} = R(-1 + \cos 2\alpha)$$

b) (30 pts) The bob is released at rest from an initial angle α_0. Find the period of oscillation T *without* using the equations of motion. You might **not** assume that α_0 is small.

c) (20 pts) Write the Lagrangian of the system $\mathcal{L}(\alpha)$;

d) (25 ptst) Find the equation of motion for $\sin\alpha$ (**not** for α).

Potentially useful identities

$$\int_0^a \frac{\mathrm{dx}}{1+\cos(2x)} = \frac{\tan a}{2}; \int_0^a \mathrm{dx}\sqrt{\frac{1+\cos(2\mathrm{x})}{\cos(2\mathrm{x})-\cos(2\mathrm{a})}} = \frac{\pi}{2}$$

$$\int_0^a \frac{\mathrm{dx}}{\sqrt{\cos x - \cos(a)}} \simeq \frac{\pi}{\sqrt{2}}\left(1+\frac{k^2}{4}+\mathcal{O}(k^4)\right), \text{ with } \mathrm{k} \equiv \sin(\mathrm{a}/2)$$

$$\int_a^\pi \mathrm{dx}\frac{\sin(\mathrm{x}/2)}{\sqrt{\cos(\mathrm{a})-\cos(\mathrm{x})}} = \frac{\pi}{\sqrt{2}}; \int_{\mathrm{a}}^\pi \mathrm{dx}\ \mathrm{x}\cos(\mathrm{x}) = \frac{1}{2}\left(-\mathrm{a}^2+\pi^2-2\sin\mathrm{a}\right)$$

$$\int \mathrm{d}x\sqrt{1-x^2} = \frac{1}{2}\left[x\sqrt{1-x^2}+\arcsin x\right]; \int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x$$

$$\int \frac{x^2\mathrm{d}x}{\sqrt{1-x^2}} = \frac{1}{2}\left[\arcsin x - x\sqrt{1-x^2}\right]; \int \frac{x^4\mathrm{d}x}{\sqrt{1-x^2}} = \frac{1}{8}\left[3\arcsin x - x\sqrt{1-x^2}(2x^2+3)\right]$$

$$\int \mathrm{d}x\sqrt{2\left(1-\cos x\right)}\mathrm{d}x = -2\sqrt{2-2\cos x}\cot(x/2)$$

$$\int \mathrm{dx}\ \sin^2(x) = \frac{x}{2} - \frac{1}{4}\sin(2x)$$

$$\int \mathrm{dx}\ \cos(x)\sin(x) = -\frac{1}{2}\cos^2 x$$

$$2\cos^2 x = 1+\cos 2x$$

$$2\sin^2 x = 1-\cos 2x$$

$$\tan x = \frac{1-\cos 2x}{\sin 2x}$$

$$\sin(a\pm b) = \sin a\cos b \pm \cos a \sin b$$

$$\cos(a\pm b) = \cos a\cos b \mp \sin a \sin b$$

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS

Academic Programs
Room 4-315

Phone: (617) 253-4841
Fax: (617) 258-8319

DOCTORAL GENERAL EXAMINATION

WRITTEN EXAM - ELECTRICITY AND MAGNETISM

Friday, August 27, 2021

DURATION: 75 MINUTES

1. This examination has two problems. Read both problems carefully before making your choice. Submit ONLY one problem. IF YOU SUBMIT MORE THAN ONE PROBLEM FROM A SECTION, BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.
2. If you decide at this time not to take this section of the exam, please inform the faculty proctor. ONCE YOU BEGIN THE EXAM, IT WILL BE COUNTED.
3. Calculators may not be used.
4. No books or reference materials may be used.

Electromagnetism 1: Radiation from a Spherical Antenna

This problem is stated in SI units.

Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a very small insulating gap.

(a) Assume the upper half of the sphere is at potential $+V$ and the lower half is at potential $-V$. Find the first two non-trivial terms in the expansion (for large r) of the potential $\Phi(r,\theta)$ in the exterior region $r > R$.

(20 points)

Now an alternating potential is applied between the two halves so that the potentials on each half are $\pm V\cos\omega t$.

(b) What is the value of the potential $\Phi(r,\theta,t)$ in the near zone ($R \ll r \ll \lambda = 2\pi c/\omega$) ? Give only the leading term. Noting that the potential for an electric dipole is $\Phi = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{x}}{|\vec{x}|^3}$, identify a time varying dipole moment $\vec{p}(t)$ and give its value.

(20 points)

(c) In the long wavelength limit find the radiation fields ($\omega = ck$)

$$\vec{H} = \frac{ck^2}{4\pi}(\hat{n}\times\vec{p})\frac{e^{ikr}}{r}, \qquad \vec{E} = -\sqrt{\frac{\mu_0}{\epsilon_0}}\hat{n}\times\vec{H},$$

the angular distribution of radiated power $\frac{dP}{d\Omega}$, and the total radiated power .
HINT: Consider the interpretation of the Poynting vector $\vec{S} = \vec{E}\times\vec{H}^*$.

(50 points)

(d) Does this antenna also radiate as an electric quadrupole? Explain why or why not. (10 point)

In the above the time-dependent quantities are $H(\vec{x},t) = \mathrm{Re}\left(\vec{H}(\vec{x})e^{-i\omega t}\right)$ and similarly for $\vec{p}$ and $\vec{E}$. The unit vector $\hat{n}$ is in the direction of observation.

Useful formulae for Legendre polynomials: $P_\ell(x)$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right), \quad P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right),$$

$$\int_{-1}^{+1} P_{\ell'}(x)P_\ell(x)dx = \frac{2}{2\ell+1}\delta_{\ell'\ell}.$$

Electromagnetism 2: Electron-neutron scattering

At very low-energy, the Born approximation to the amplitude for an electron scattering from a neutron is given by

$$A = -\frac{m}{\pi\hbar^2}\int d^3r V_E(\mathbf{r}),$$

where m is the electron mass and the potential energy of interaction between the electron and the neutron,

$$V_E(\mathbf{r}) = \int d^3s \rho_e(\mathbf{s})\varphi_N(\mathbf{r}-\mathbf{s})$$

is given in terms of the charge density of the electron, $\rho_e(\mathbf{s})$, and the electrostatic potential of the neutron, $\varphi_N(\mathbf{s})$.

(a) Show that an infinitesimal displacement of the electron charge density by a distance $\delta\mathbf{s}$ induces a infinitesimal change in the potential energy V_E such that it leads to a Coulomb force

$$\mathbf{F} = -\frac{\delta V}{\delta \mathbf{s}} = \int d^3r \rho_e(\mathbf{r})\mathbf{E}_N(\mathbf{r}),$$

where $\mathbf{E}_N(\mathbf{r})$ is the electric field produced by the neutron.
HINT: Consider first the infinitesimal change in the electron charge density $\delta\rho_e(\mathbf{r})$.

(40 points)

(b) Show that the interaction potential energy can instead be written in terms of the charge distribution of the neutron, $\rho_N(\mathbf{r})$, and the electrostatic potential due to the (pointlike) electron, $\varphi_e(\mathbf{r})$.

(20 points)

(c) Suppose that $\rho_N(\mathbf{s}) = \rho_N(|\mathbf{s}|)$ is spherically symmetric and that $\varphi_e(\mathbf{s})$ varies slowly over the region of space occupied by the neutron's charge distribution. Show that the scattering amplitude can be written as

$$\int d^3r V_E(\mathbf{r}) = -\frac{1}{6}\frac{e}{\epsilon_0}\int d^3s |\mathbf{s}|^2 \rho_N(\mathbf{s}).$$

HINT: The electron charge density satisfies Poisson's equation.

(35 points)

(d) What information does a measurement of the low energy cross section reveal about the neutron? That is, what is a physical interpretation of integral on the RHS of the equation in part (c) above?

(5 points)

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS

Academic Programs
Room 4-315

Phone: (617) 253-4841
Fax: (617) 258-8319

DOCTORAL GENERAL EXAMINATION

WRITTEN EXAM - STATISTICAL MECHANICS

Tuesday, August 31, 2021

DURATION: 75 MINUTES

1. This examination has two problems. Read both problems carefully before making your choice. Submit <u>ONLY</u> one problem. IF YOU SUBMIT MORE THAN ONE PROBLEM FROM A SECTION, BOTH WILL BE GRADED, AND THE PROBLEM WITH THE <u>LOWER SCORE</u> WILL BE COUNTED.
2. If you decide at this time not to take this section of the exam, please inform the faculty proctor. ONCE YOU BEGIN THE EXAM, IT WILL BE COUNTED.
3. Calculators may not be used.
4. No books or reference materials may be used.

Statistical Mechanics 1: Stoner Magnetism

The conduction electrons in a metal can be treated as a gas of fermions with spin 1/2 and density $n = N/V$. Coulomb repulsion favors two-electron wavefunctions whose spatial part is antisymmetric with respect to exchange of the two electrons' position, thus keeping the electrons apart. Because of the full (spatial and spin) antisymmetry of fermionic wave functions, this mutual repulsion may be approximated by an effective spin-spin coupling which favors states with parallel spins. In this simple approximation, the net effect is described by an interaction energy

$$U = \alpha \frac{N_+ N_-}{V}, \tag{1}$$

where N_+ and $N_- = N - N_+$ are the numbers of electrons with up and down spins, respectively, and V is the volume. (The parameter α is related to the scattering length a by $\alpha = 4\pi\hbar^2 a/m$ and can be treated as a phenomenological constant here.)

(a) The ground state has two Fermi seas filled by the spin up and spin down electrons. Express the corresponding Fermi wavefectors $k_{F\pm}$ in terms of the densities $n_\pm = N_\pm/V$. (*20 pts*)

(b) Calculate the kinetic energy density of the ground state as a function of the densities $n_\pm$ and fundamental constants. (*20 pts*)

(c) Assuming small deviations $n_\pm = \frac{n}{2}(1 \pm \delta)$ from the symmetric state, expand the kinetic energy to fourth order in δ. (*20 pts*)

(d) Express the spin-spin interaction density U/V in terms of n and δ. Find the critical value of α_c such that for $\alpha > \alpha_c$ the electron gas can lower its total energy by spontaneously developing a magnetization. (This is known as the Stoner Instability.) (*20 pts*)

(e) What is the behavior of the magnetization for α above and below α_c? Sketch the magnetization as a function of α. (*20 pts*)

Reminder: The Taylor series of a function $f(x)$ about a point $x = a$ is $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, where $f^{(n)}(a)$ is the nth derivative of f evaluated at a.

Statistical Mechanics 2: A Critical Point

The pressure P of a gas is related to its density $n = N/V$ and temperature T by the expression

$$P = k_B T n - \frac{b}{2} n^2 + \frac{c}{6} n^3, \tag{1}$$

where b and c are positive, temperature independent constants.

(a) Determine the critical temperature T_c below which the above equation must be invalid and the corresponding density n_c and pressure P_c of the critical point. Evaluate the ratio $k_B T_c n_c / P_c$. *(20 pts)*

(b) Calculate the isothermal compressibility $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)|_T$ and sketch its behavior as a function of T for $n = n_c$. *(20 pts)*

(c) Give an expression for $(P - P_c)$ as a function of $(n - n_c)$ on the critical isotherm. *(20 pts)*

(d) The instability in the isotherms for $T < T_c$ is avoided by phase separation into a liquid of density n_+ and gas of density n_-. For temperatures close to T_c, these densities behave as $n_\pm \approx n_c(1 \pm \delta)$. Using a Maxwell construction, or otherwise, find an implicit equation for $\delta(T)$. *Hint:* Along an isotherm, variations of the chemical potential obey $d\mu = dP/n$. *(40 pts)*

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS

Academic Programs
Room 4-315

Phone: (617) 253-4841
Fax: (617) 258-8319

DOCTORAL GENERAL EXAMINATION

WRITTEN EXAM - QUANTUM MECHANICS

Monday, August 30, 2021

DURATION: 75 MINUTES

1. This examination has two problems. Read both problems carefully before making your choice. Submit ONLY one problem. IF YOU SUBMIT MORE THAN ONE PROBLEM FROM A SECTION, BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.
2. If you decide at this time not to take this section of the exam, please inform the faculty proctor. ONCE YOU BEGIN THE EXAM, IT WILL BE COUNTED.
3. Calculators may not be used.
4. No books or reference materials may be used.

Quantum Mechanics 1: Coherent states

Here we study the properties of coherent states of the harmonic oscillator, defined as eigenstates of the annihilation operator

$$\hat{a}|\overline{\alpha}\rangle = \alpha|\overline{\alpha}\rangle,$$

with eigenvalue $\alpha \in \mathbb{C}$ and having the representation

$$|\overline{\alpha}\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \,,$$

where $|n\rangle$ are the usual eigenstates of the number operator.

(a) (*20 pts*) Calculate the inner product, $\langle \overline{\alpha} \,|\overline{\beta}\, \rangle$, of two coherent states and discuss whether the pure coherent states above are orthogonal for any choices of α and β.

Reminder: $\sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x$.

(b) (*40 pts*) Consider the following superpositions of two coherent states $|\overline{\beta}\rangle$ and $|\overline{-\beta}\rangle$ for $\beta \in \mathbb{R}$

$$|\psi_\pm\rangle = \frac{\mathcal{N}_\pm(\beta)}{\sqrt{2}} \left(|\overline{\beta}\rangle \pm |\overline{-\beta}\rangle\right)$$

(i) Find $\mathcal{N}_\pm(\beta)$ such that these states are unit normalised and show that $\mathcal{N}_+(|\beta| \to \infty) = 1$.

(ii) Show that $|\psi_+\rangle$ and $|\psi_-\rangle$ are orthogonal.

(iii) Obtain the number distribution of these states, $P_\pm(n) = |\, \langle n \,|\psi_\pm \rangle \,|^2$.

(c) (*40 pts*) Consider the three operators

$$\hat{K}_1 = \frac{1}{2}(\hat{a}^{\dagger 2} + \hat{a}^2), \qquad \hat{K}_2 = \frac{1}{2i}(\hat{a}^{\dagger 2} - \hat{a}^2), \qquad \hat{K}_3 = \frac{1}{2}\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right).$$

(i) Calculate the commutator of $\hat{K}_1$ and $\hat{K}_2$, expressing it in terms of $\hat{K}_3$.

(ii) Use this relation to write down a lower bound on the product of uncertainties $\Delta K_1 \Delta K_2$ in an arbitrary state $|\Psi\rangle$. Ascertain by explicit calculation whether this bound is saturated (that is, becomes an equality) in a coherent state $|\overline{\gamma}\rangle$ with $\gamma \in \mathbb{C}$.

Hint: To aid in this, note that

$$[\hat{a}^2, \hat{a}^{\dagger 2}] = 4\hat{a}^\dagger \hat{a} + 2$$

$$\hat{K}_1^2 = \frac{1}{4}(\hat{a}^{\dagger 4} + \hat{a}^4 + 2\hat{a}^{\dagger 2}\hat{a}^2 + 4\hat{a}^\dagger \hat{a} + 2)\,,$$

$$\hat{K}_2^2 = -\frac{1}{4}(\hat{a}^{\dagger 4} + \hat{a}^4 - 2\hat{a}^{\dagger 2}\hat{a}^2 - 4\hat{a}^\dagger \hat{a} - 2)\,.$$

Quantum Mechanics 2: Particles in a Uniform Magnetic Field

The Hamiltonian for an electron moving in a magnetic field can be written as:

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 - \vec{\mu}\cdot\vec{B},$$

where m is the electron mass, $q < 0$ is its charge, and $\vec{\mu} = \frac{q}{mc}\vec{s}$ is its magnetic dipole moment with $\vec{s}$ being the spin operator.

Here we consider the case of an electron moving in a uniform magnetic field directed along the z-axis (i.e. $\vec{B} = B\hat{k}$), described by a vector potential $\vec{A} = -By\hat{\imath}$. This system can be described by a two-component Schrodinger wave function:

$$\psi(\vec{x}, t) \equiv \begin{pmatrix} \psi_+(\vec{x}, t) \\ \psi_-(\vec{x}, t) \end{pmatrix},$$

where ψ_+ and ψ_- refer to the amplitudes for spin-up and spin-down, respectively, along the z-axis. The time-independent Schrodinger equation then has solutions of the form:

$$\psi(\vec{x}) = e^{\frac{i}{\hbar}(p_x + p_z z)}\phi(y)\chi_\pm,$$

with $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(a) (*20 pts*) Using $\vec{p} = -i\hbar\vec{\nabla}$ and $\vec{s} = \frac{1}{2}\hbar\vec{\sigma}$ (where σ is the Pauli matrix), use the Schrodinger equation to obtain the equation that must be obeyed by $\phi(y)$ (you do not need to solve it).

(b) (*20 pts*) Find the complete set of energy eigenvalues.

Note: (A) defining $y_0 \equiv -\frac{p_x c}{qB}$ could simplify the relevant expressions. (B) you are *not* required to also find the corresponding eigenfunctions.

(c) (*25 pts*) Electroweak gauge invariance allows replacing the vector potential $\vec{A}$ by:

$$\vec{A}' \equiv \vec{A}(\vec{x}) + \vec{\nabla}\Lambda(\vec{x}),$$

where $\Lambda(\vec{x})$ is an arbitrary scalar function. When such gauge transformation is carried our, the Schrodinger wave function $\psi(\vec{x}, t)$ must be replaced by a transformed wave function $\psi'(\vec{x}, t)$. The formalism is set up such that $\left(\vec{p} - \frac{q}{c}\vec{A}\right)$ is gauge-covariant and we can thus construct the transformed wave function $\psi'(\vec{x}, t)$ in terms of $\psi(\vec{x}, t)$ and $\Lambda(\vec{x})$ as $\psi'(\vec{x}) = e^{\frac{i}{\hbar}\frac{q}{c}\Lambda(\vec{x})}\psi(\vec{x})$.

Derive the time-independent Schrodinger equation in the new gauge and explain its relation to the corresponding equation in the original gauge.

(d) (*35 pts*) The (unnormalized) ground state wave function can be written in the form $\psi(\vec{x}) = e^{\frac{i}{\hbar}(p_x + p_z z)}\phi(y)\chi_{\pm}$, described above, with

$$\phi(y) = e^{-\frac{|q|B}{2\hbar c}(y-y_0)^2},$$

where $y_0 \equiv -\frac{p_x c}{qB}$. This wave function is localized in the y-dimension, but had an infinite extent in the x-direction. Since the physics of this problem is invariant under rotations about the z-axis, it must also be possible to find a solution to the time-independent Schrodinger equation of part (a) which are localized in the x-dimension but have an infinite extent in the y-direction.

Write down an explicit expression for such a wave function.

Hint: While you are expected to use the same gauge for the vector potential as above ($\vec{A} = -By\hat{\imath}$), it is useful to start by considering the corresponding Schrodinger equation for the modified gauge $\vec{A}' = Bx\hat{\jmath}$, finding $\phi'(x)$ and the relevant $\Lambda(x)$ function so that you can transform back to the original (un-primed) gauge to obtain the requested $\psi(x)$ wave function.