Mechanics

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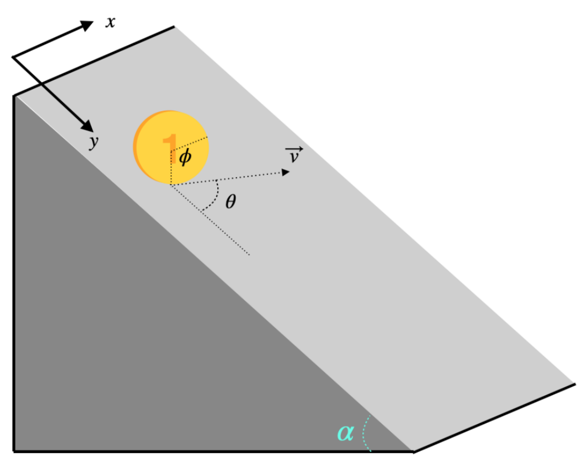
1 Problem 1

Solution by Rikab Gambhir, with corrections and notes by Daniel Mark.

Skateboarding was one of the new disciplines introduced at the Tokyo 2020 Olympic games. The winner of the gold medal is getting ready for their mechanics exam by letting the medal roll along the skateboard ramp and analyzing its motion.

To simplify the math, they, and us, will assume that the medal is a thin disk of constant density, with total mass M and radius R (that is: we entirely neglect its thickness). We define coordinates x and y on the (flat) surface of the ramp as in the Figure.



We will use an angle ϕ to measure rotations around the medal axis and θ to measure rotations around an axis perpendicular to the ramp and passing through the contact point (see Figure). We define θ such that $\theta = 0$ if the instantaneous velocity v of the medal is along the $+y$ direction. We call α the angle that the incline makes with the horizontal and take the acceleration of gravity to be g, directed downward. We will assume that the medal is constrained such that its plane is always perpendicular to the ramp, and that it rolls without slipping.

1.1 Part a

(10 pts) Calculate all the non-zero compontents of the moment of inertia tensor for the medal relative to a coordinate system whose origin is at the center of the medal, with the Z axis perpendicular to the medal and the X and Y axes in the plane of the medal;

We are tasked to find the Inertia tensor, I_{ij}, for a thin disk. Here is the definition of the inertia tensor:

$$I_{xx} = \sum_i (y_i^2 + z_i^2) m_i^2 = \int (y^2 + z^2) dm \tag{1.1}$$

$$I_{yy} = \sum_i (z_i^2 + x_i^2) m_i^2 = \int (z^2 + x^2) dm \tag{1.2}$$

$$I_{zz} = \sum_i (x_i^2 + y_i^2) m_i^2 = \int (x^2 + y^2) dm \tag{1.3}$$

$$I_{xy} = I_{yx} = -\sum_i (x_i y_i) m_i^2 = -\int (xy) dm \tag{1.4}$$

$$I_{yz} = I_{zy} = -\sum_i (y_i z_i) m_i^2 = -\int (yz) dm \tag{1.5}$$

$$I_{zx} = I_{xz} = -\sum_i (z_i x_i) m_i^2 = -\int (zx) dm \tag{1.6}$$

For these sorts of problems, the gimmick is to rewrite dm in terms of dx, dy, and dz. The disk has a mass density of $\sigma = \frac{M}{\pi R^2}$, so we can write a mass element of the disk as $dm = \sigma r dr d\phi$, in polar coordinates along x and y. Now we can calculate:

$$I_{zz} = \int dm (x^2 + y^2) \tag{1.7}$$

$$= \sigma \int_0^R dr \int_0^{2\pi} d\phi r^3 \tag{1.8}$$

$$= \frac{1}{2} M R^2 \tag{1.9}$$

Note that for a thin disc, z = 0.

$$I_{xx} = \int dm (y^2 + z^2) \tag{1.10}$$

$$= \sigma \int_0^R dr \int_0^{2\pi} d\phi r^3 \sin^2(\phi) \tag{1.11}$$

$$= \frac{1}{4} M R^2 \tag{1.12}$$

It is easy to see that I_{yy} will be the same as I_{xx}. Since $z = 0$, that means I_{zx} and I_{yz} are immediately zero. Finally, I_{xy} is zero since $\sin(\phi)\cos(\phi)$ is zero over one period.

Note (DM): Alternatively, we can simply recall the result that $I_{zz} = \frac{1}{2} M R^2$, and also use the *perpendicular axis theorem*: for a 2d object lying in the xy plane (or a 2d object stretched and extended along its axis z), $I_{zz} = I_{xx} + I_{yy}$. In our case, by symmetry, $I_{xx} = I_{yy} = I_{zz}/2$.

1.2 Part b

(10 pts) Write down two independent constraints that might relate some or all of ϕ, θ, x, y and/or their time derivatives;

Our constraint is rolling without slipping, meaning that the velocity of the disk on the plane must match R times its angular velocity:

$$|v| = R\dot{\phi} \tag{1.13}$$

We can write this as a constraint on the x and y components of the velocity:

$$\dot{x} = R\sin(\theta)\dot{\phi} \tag{1.14}$$

$$\dot{y} = R\cos(\theta)\dot{\phi} \tag{1.15}$$

Note that these constraints are *nonholonomic*, meaning that cannot be written as $f(q) = 0$, since they depend on velocities. This will make the usual trick of having Lagrange multipliers of the form $\lambda f(q)$ in the Lagrangian a bit tricky. For use later, let us rewrite these constraints in the following slick form:

$$f_1 \cdot dq = 0;\;\; f_1 = (1, 0, -R\sin(\theta), 0) \tag{1.16}$$

$$f_2 \cdot dq = 0;\;\; f_2 = (0, 1, -R\cos(\theta), 0) \tag{1.17}$$

$$dq = (dx, dy, d\phi, d\theta) \tag{1.18}$$

1.3 Part c

(20 pts) Write the Lagrangian of the system in terms of some or all of ϕ, θ, x, y and/or their time derivatives;

Our task is to write the Lagrangian. Lets start with $\mathcal{L} = T - V$ and work from there. The gravitational potential is $V = mgh = -mgy\sin(\alpha)$. The kinetic energy is T $= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_{zz}\dot{\phi}^2 + \frac{1}{2}I_{xx}\dot{\theta}^2$

So we can write:

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_{zz}\dot{\phi}^2 + \frac{1}{2}I_{xx}\dot{\theta}^2 \\ &+ mgy\sin(\alpha) \\ &- \lambda_1(\dot{x} - R\sin(\theta)\dot{\phi}) \\ &- \lambda_2(\dot{y} - R\cos(\theta)\dot{\phi})\end{aligned} \tag{1.19}$$

[I am actually not sure if it is ok to treat the constraints using regular Lagrange multipliers like this. Nevertheless we will press on. –Rikab]

Comment (DM): Rikab's approach in part d essentially rederives the Euler-Lagrange equation with constraints, i.e. we never actually use the $\mathcal{L}$ in Eq. 1.19)

1.4 Part d

(35 pts) Find the equations of motion for $\theta(t)$ and $\phi(t)$. Your answer cannot contain x or y;

Rather than using the EL equations directly with non-holonomic constraints (they are very messy, with λ's and $\dot{\lambda}$'s flying everywhere), we are going to go back to basics to incorporate the constraints in a slick way. Remember that the equations of motion are the special path $q(t)$ such that to first order,

$$S[q(t)] = \int dt\mathcal{L}(q, \dot{q}) \tag{1.20}$$

is unchanged under $q \to q + \delta q$. Usually, we let δq be anything, but for a constrained system, the only allowed variations have to satisfy the constraints! That is, $f_i \cdot \delta q = 0$ for each constraint i, using our notation from earlier. So,

$$\begin{aligned}
\delta S &= 0 && (1.21)\\
&= \int dt\delta\mathcal{L} - \sum_i \lambda_i f_i \cdot \delta q && (1.22)\\
&= \int dt \left[\frac{\partial\mathcal{L}}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}}\right]\delta q - \sum_i \lambda_i f_i \cdot \delta q && (1.23)\\
&\downarrow\\
\frac{\partial\mathcal{L}}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}} &= \sum_i \lambda_i f_i && (1.24)
\end{aligned}$$

Now we can write EOM (using the above Lagrangian without the two extra terms):

$$\begin{aligned}
&x : -m\ddot{x} = \lambda_1 && (1.25)\\
&y : mg\sin(\alpha) - m\ddot{y} = \lambda_2 && (1.26)\\
&\phi : -I_{zz}\ddot{\phi} = -\lambda_1 R\sin(\theta) - \lambda_2 R\cos(\theta) && (1.27)\\
&\theta : -I_{xx}\ddot{\theta} = 0 && (1.28)
\end{aligned}$$

Note that the EOM for θ makes sense. There are no torques perpendicular to the inclined plane, so angular momentum along that axis should be conserved. In order to write the EOM for ϕ without references to x or y, we can plug in $\ddot{x} = R(\sin(\theta)\ddot{\phi} + \cos(\theta)\dot{\phi}\dot{\theta})$ and $\ddot{y} = R(\cos(\theta)\ddot{\phi} - \sin(\theta)\dot{\phi}\dot{\theta})$

$$\begin{aligned}-I_{zz}\ddot{\phi} &= mR^2(\sin(\theta)\ddot{\phi} + \cos(\theta)\dot{\phi}\dot{\theta})\sin(\theta) \\ &+ mR^2(\cos(\theta)\ddot{\phi} - \sin(\theta)\dot{\phi}\dot{\theta})\cos(\theta) \\ &- mgR\cos(\theta)\sin(\alpha)\end{aligned} \tag{1.29}$$

Simplify...

$$(I_{zz} + mR^2)\ddot{\phi} = mgR\cos(\theta)\sin(\alpha) \tag{1.30}$$

$$\frac{1}{\Omega^2}\ddot{\phi} = \cos(\theta)\ ,\ \Omega^2 = \frac{2g}{3R}\sin\alpha \tag{1.31}$$

Note again that this EOM makes sense! If α were zero, we should not expect ϕ to accelerate. The disc can also only roll due to gravity when $\theta = 0$ and not when $\theta = \pi/2$, so the cosine makes sense. The value of the acceleration: $\Omega^2 = \frac{2g}{3R}\sin\alpha = \frac{1}{1+1/2}\frac{g}{R}\sin\alpha$, is also the correct value for a disk rolling without slipping.

1.5 Part e

(25 pts) Find explicit solutions for $x(t)$ and $y(t)$. Call $\omega \equiv d\theta/dt|_{t=0} \neq 0$. You can use the initial conditions: $x(t = 0) = y(t = 0) = \phi(t = 0) = \theta(t = 0) = d\phi/dt|_{t=0} = 0$

Since θ does not accelerate, we know $\theta = \omega t$. Then we can plug in to the EOM for ϕ.

$$\frac{1}{\Omega^2}\ddot{\phi} = \cos\omega t \tag{1.32}$$

Integrate twice to get the answer, and place constants to ensure the initial conditions are met:

$$\phi(t) = \frac{\Omega^2}{\omega^2}(1 - \cos\omega t) \tag{1.33}$$

Use our λ_1 and λ_2 from earlier to plug into the EOM for x and y:

$$\ddot{x} = R(\sin(\theta)\ddot{\phi} + \cos(\theta)\dot{\phi}\dot{\theta}) \tag{1.34}$$

$$= R\Omega^2(\sin(\omega t)\cos(\omega t) + \cos(\omega t)\sin(\omega t)) \tag{1.35}$$

$$= R\Omega^2\sin(2\omega t) \tag{1.36}$$

Integrating...

$$\dot{x} = -R\frac{\Omega^2}{2\omega}\cos 2\omega t + A;\ \ A = R\frac{\Omega^2}{2\omega} \tag{1.37}$$

$$x = -R\frac{\Omega^2}{4\omega^2}\sin 2\omega t + At \tag{1.38}$$

Similarly for y...

$$y = R\frac{\Omega^2}{4\omega^2}(\cos 2\omega t - 1) + \frac{1}{2}g\sin(\alpha)t^2 \tag{1.39}$$

Check for reasonableness: if $\alpha = 0$, the disc should not move at all, and indeed the EOM become zero. When we take $\omega \to 0$, being careful with limits, we see that $x = 0, y \sim t^2$, and $\phi \sim t^2$, as we should expect! More concretely, when $\omega \to 0$, we expand the cosine and obtain:

$$\phi(t) = \frac{1}{2}\left(\frac{2g}{3R}\sin\alpha\right)t^2 \tag{1.40}$$

and

$$x(t) = 0 \tag{1.41}$$

$$y(t) = \frac{1}{2}\left(\frac{2g}{3}\sin\alpha\right)t^2 \tag{1.42}$$

Note (DM): I tried to solve this problem from a Newtonian perspective by: 1. calculating the friction force on the disk, 2. calculating the resulting torques in the *body axes*, and 3. using Euler's equations for rigid body motion to obtain equations of motion for θ and ϕ. The approach is promising, but there is an internal inconsistency that I was unable to resolve, indicating that my handling of the friction forces is incorrect. (Determining the friction force is a little delicate because of the non-inertial *body* frame, and is just about as complicated as the above Lagrangian method.)

2 Problem 2: Two pendula

Solution by Daniel Mark

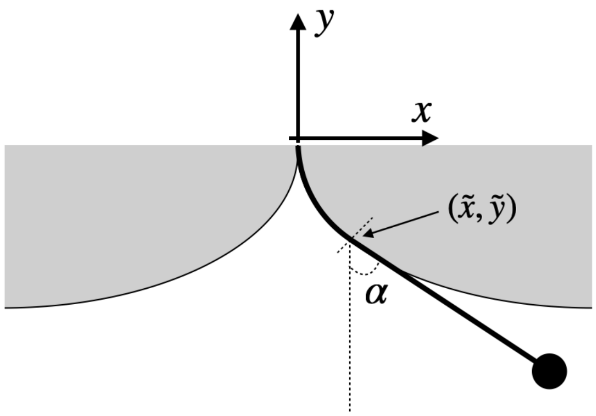
[Throughout this problem you can assume a constant gravitational field, with acceleration $\vec{g} = -g\hat{y}$, with $g > 0$.]

Consider first a simple pendulum with mass M hanging at the end of an inextensible and massless string of length $L = 4R$. The pendulum is released at rest from an initial angle γ_0, which you may not consider small.

2.1 Part a

(25 pts) Find the period of oscillation and explicitly comment on whether it increases, stays constant, or decreases with γ_0. Now consider a different pendulum, also of mass M hanging at the end of an inextensible and massless string of length $L = 4R$. The ceiling that the

string of this second pendulum is fixed to is not flat. Instead, it has the shape shown in figure:



As the pendulum swings, the string can wrap partially around the ceiling. We call α the angle between the part of string that is not wrapped around the ceiling and the vertical direction. The symmetric bump in the ceiling is constructed such that the length of the string that is touching the ceiling — $\ell(\alpha)$ — is a simple function of α:

$$\ell(\alpha) = 4R(1 - \cos\alpha) \ , \tag{2.1}$$

while the (x, y) coordinates of the point where the string leaves the ceiling are:

$$\tilde{x} = R[2\alpha\sin(2\alpha)]; \tilde{y} = R[1 + \cos(2\alpha)] \tag{2.2}$$

Let θ be the angle of the pendulum. We apply conservation of energy:

$$E = \frac{1}{2}mL^2\dot{\theta}^2 - mgL\cos\theta \tag{2.3}$$

So

$$\frac{1}{2}mL^2\dot{\theta}^2 - mgL\cos\theta = -mgL\cos\gamma_0 \tag{2.4}$$

$$\Longrightarrow \frac{d\theta}{dt} = \pm\sqrt{2\frac{g}{L}\left[\cos\theta - \cos\gamma_0\right]} \tag{2.5}$$

$$\Longrightarrow \boxed{T = \int dt = 4\sqrt{\frac{L}{2g}}\int_0^{\gamma_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\gamma_0}}} \tag{2.6}$$

$$\approx 4\pi\sqrt{\frac{R}{g}}\left[1 + \frac{\sin^2(\gamma_0/2)}{4} + \cdots\right] \tag{2.7}$$

The factor of 4 in Eq. (2.6) is because the motion from $\theta = 0$ to $\theta = \gamma$ is a *quarter* of one oscillation.

The second term in Eq. (2.7) is always positive and increases with γ_0. Therefore, the period of the pendulum **increases** with γ_0.

2.2 Part b

(30 pts) The bob is released at rest from an initial angle α_0. Find the period of oscillation T without using the equations of motion. You might not assume that α_0 is small.

We find expressions for the coordinates (X, Y) of the bob: With $\ell(\alpha) = 4R(1 - \cos\alpha)$, the length of the string is $4R\cos\alpha$. Therefore

$$X = \tilde{x} + 4R\cos\alpha\sin\alpha = R(2\alpha + \sin(2\alpha)) \tag{2.8}$$
$$Y = \tilde{y} - 4R\cos^2\alpha = -R(3 + \cos(2\alpha)) \tag{2.9}$$

So the kinetic energy of the bob is:

$$T = \frac{1}{2}m\left(\dot{X}^2 + \dot{Y}^2\right) \tag{2.10}$$
$$= \frac{1}{2}m\ 4R^2\dot{\alpha}^2\left((1+\cos(2\alpha))^2 + \sin^2(2\alpha)\right) \tag{2.11}$$
$$= 4mR^2\dot{\alpha}^2\left(1+\cos(2\alpha)\right) = 8mR^2\dot{\alpha}^2\cos(\alpha)^2 \tag{2.12}$$

The potential energy of the bob is:

$$V = mgY = -mgR(3+\cos(2\alpha)) \tag{2.13}$$
$$= -2mgR(1+\cos^2(\alpha)) \tag{2.14}$$

As in (a) we obtain:

$$8mR^2\dot{\alpha}^2\cos^2(\alpha) - 2mgR(1+\cos^2(\alpha)) = -2mgR(1+\cos^2(\alpha_0)) \tag{2.15}$$
$$\implies \dot{\alpha} = \sqrt{\frac{g}{4R}}\frac{\sqrt{\cos^2(\alpha) - \cos^2(\alpha_0)}}{\cos(\alpha)} \tag{2.16}$$
$$= \sqrt{\frac{g}{4R}}\sqrt{\frac{\cos(2\alpha) - \cos(2\alpha_0)}{1+\cos(2\alpha)}} \tag{2.17}$$
$$\implies T = \int dt = 4\sqrt{\frac{4R}{g}}\int_0^{\alpha_0} d\alpha\sqrt{\frac{1+\cos(2\alpha)}{\cos(2\alpha)-\cos(2\alpha_0)}} \tag{2.18}$$
$$= \boxed{4\pi\sqrt{\frac{R}{g}}} \tag{2.19}$$

Which is the small-angle period T of a normal pendulum, see (a).

2.3 Part c

(20 pts) Write the Lagrangian of the system $\mathcal{L}(\alpha)$;

From above, the Lagrangian is:

$$\mathcal{L}(\alpha) = T - V \tag{2.20}$$

$$= \boxed{8mR^2\dot{\alpha}^2\cos(\alpha)^2 + 2mgR(1+\cos^2(\alpha))} \tag{2.21}$$

Remark: From the point allocation, there is perhaps a different way of tackling this problem (e.g. by expressing $\dot{\alpha}$ in terms of the instantaneous length of the pendulum). However, I believe the method presented to be the most foolproof.

2.4 Part d

(25 pts) Find the equation of motion for $\sin\alpha$ (**not** for α). Re-expressing the Lagrangian in terms of $\sin\alpha$:

$$\mathcal{L}(\sin\alpha) = 8mR^2\left(\frac{d\sin\alpha}{dt}\right)^2 + 2mgR(2-\sin^2\alpha) \tag{2.22}$$

This is, up to an additive constant, the Langrangian for a simple harmonic oscillator. Either by inspection or by using the Euler-Lagrange equation, we obtain:

$$16mR^2\frac{d^2}{dt^2}\sin\alpha = -4mgR\sin\alpha \tag{2.23}$$

$$\implies \boxed{\frac{d^2}{dt^2}\sin\alpha = -\frac{g}{4R}\sin\alpha} \tag{2.24}$$

$$\implies \sin\alpha = \sin\alpha_0\cos\left(\sqrt{\frac{g}{4R}}t\right) \tag{2.25}$$

We have simple harmonic motion in the variable $\sin\alpha$.