# MASSACHUSETTS INSTITUTE OF TECHNOLOGY

DEPARTMENT OF PHYSICS

SEPTEMBER 5, 1997

DOCTORAL GENERAL EXAMINATION

PART II

FIVE HOURS

## INSTRUCTIONS

1. This examination consists of four sections with two problems in each: Mechanics, Electricity & Magnetism, Statistical Mechanics, and Quantum Mechanics. It is advisable to read both problems in each section carefully before making your choice. SUBMIT ONLY ONE PROBLEM FROM EACH SECTION.

2. Use a separate fold of paper for each problem, and write your name on each fold. Include the problem number with each solution.

3. Calculators may be used.

4. No Books or Reference Materials May Be Used.

## CLASSICAL MECHANICS I

Based on a reasonable set of assumptions, calculate the amount of fuel (in liters) required by a fully loaded Boeing 747 to fly from London to New York. Justify all assumptions and estimate the accuracy of your answer. The following information on the Boeing 747 may be useful.

DATA TABLE

747 Performance Summary

| | |
|---|---|
| Passengers | 420 (21 first, 77 business, 322 economy class) |
| Cargo | 6,025 cubic feet (170.8 cubic meters) all containers or<br>5,332 cubic feet (151.0 cubic meters) 5 pallets, 14 LD-1 containers + bulk |
| Engines (four) | * Pratt & Whitney PW4056<br>* General Electric CF6-80C2 BIF<br>* Rolls Royce RB211-524G |

Thrust (pounds) 56,000 nominal

Maximum Takeoff Weight 800,000 pounds (362,880 kg)

Options * 833,000 pounds (377,840 kg)
* 850,000 pounds (385,560 kg)
* 875,000 pounds (396,890 kg)

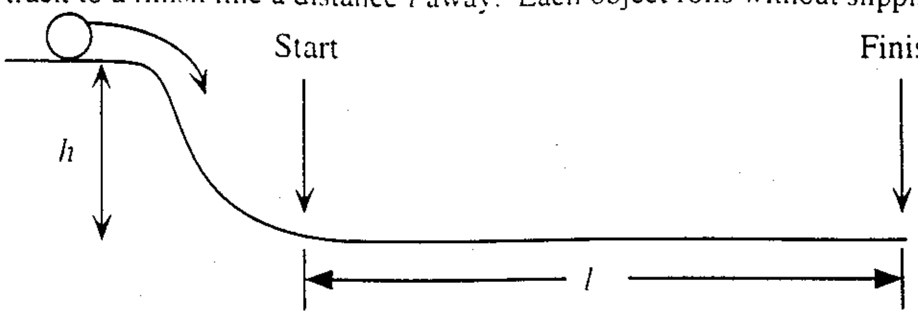
Design Range 8,290 statute miles (13,340 km)

Basic Specifications

| | |
|---|---|
| Wing Span | 211 feet 5 inches (64.44 m) |
| Overall Length | 231 feet 10.25 inches (70.66 m) |
| Tail Height | 63 feet 8 inches (19.41 m) |
| Body Width | |
| Outside | 21 feet 4 inches (6.5 m) |
| Inside | 20 feet (6.1 m) |

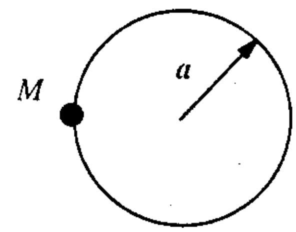
## CLASSICAL MECHANICS II

**a)** In a novel betting race, four objects, all of total mass $M$ are rolled down a ramp of height $h$ and along a track to a finish line a distance $l$ away. Each object rolls without slipping.




The objects are

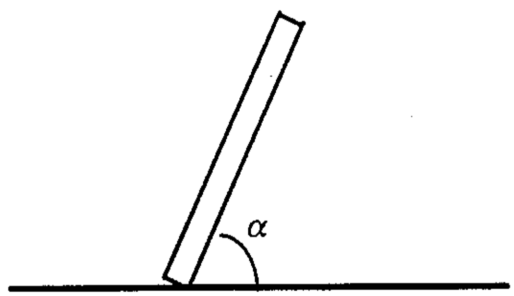
1. a uniform disk of radius $a$
2. a ring of radius $a$
3. a sphere of radius $a$
4. a ring of negligible mass with the mass $M$ located at one point as shown at right.




Object 4

Determine the order of finish for all values of $l$ and $h$.

**b)** Object 1, a uniform disk of mass $M$ and radius $a$ is rolled on a level surface. The disk is rolled so it makes an angle $\alpha$ with the surface as shown. The disk will roll in a circle of radius $R$.. Find $R$ in terms of $\alpha$ and the initial velocity.

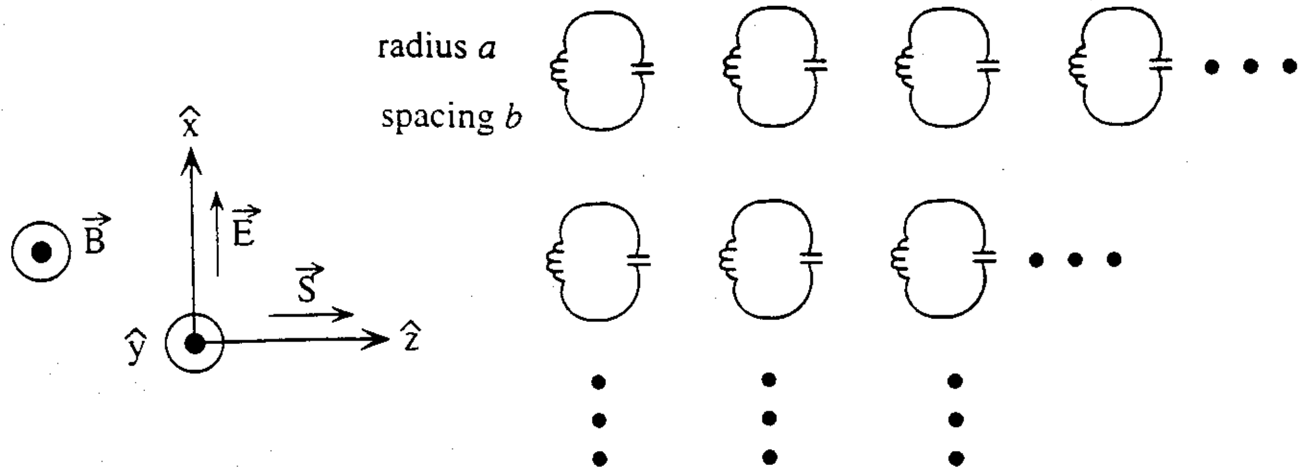



**c)** The disk is again rolled as in part b), except now there is a frictional force acting on the disk of the form $\bar{F} = -a\bar{v}$. Determine the trajectory of the disk.

## ELECTRICITY AND MAGNETISM I

An electromagnetic wave of wave length $\lambda$ and frequency $\nu$, is incident on an infinite half-space of loops with area $A$. The wavelength is long compared to the size of the loop, $\lambda >> \sqrt{A}$. The loops are a distance $l$ apart and $l >> \lambda$.

Each loop has an in inductance $L$, capacitance $C$, and resistance $R$, all in series as shown.



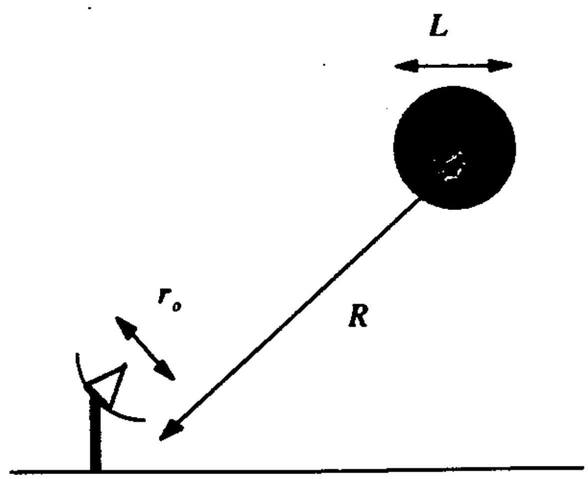

The plane of each loop in $\perp$ to $\hat{y}$ . $\vec{S} = S\hat{z}$, $\vec{S}$ is the Poynting vector.

**a)** If the incident wave is polarized $\vec{E} = E_0\hat{x}$, determine the power attenuation length. Also find the angular distribution of the reradiated wave.

**b)** Assume the loops are randomly oriented with the same spacing. Find the power attenuation length. Also find the angular distribution of the reradiated wave.

**c)** Next, assume the incident wave is unpolarized. Find the power attenuation length and the angular distribution of the reradiated wave.

**d)** The wave is generated by means of antenna mounted in a parabolic dish reflector as shown. The diameter of the dish is $r_0$. What is the opening angle of the diverging beam?

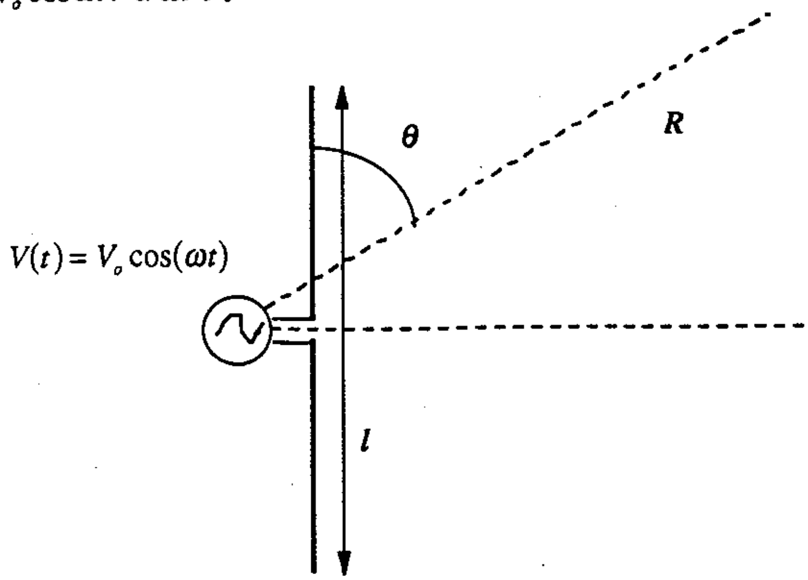



**e)** The beam from the dish reflector uniformly illuminates a finite cloud of randomly oriented loops of radius $L$, a distance $R$ from the antenna as shown. Compute the total power of the reradiated wave incident on the dish.

f) Compute the attenuation in part (a) assuming $l << \lambda$.

**ELECTRICITY AND MAGNETISM II**

A dipole transmission antenna has length $l$, capacitance $C$ and is driven with a sinusoidal potential $V(t) = V_o \cos \omega t$. $c/\omega >> l$




a) For $l << 2\pi c/\omega$, compute the radiated power density for $R >> l$ as a function of $l$ and $\theta$.

b) Compute the radiated power density for $R >> l$ as a function of $l$ and $\theta$ if $l >> 2\pi c/\omega$. What happens in the asymptotic limit as $2\pi c/\omega \to 0$.

c) From parts a and b, sketch the polar angle of the power radiation maximum as a function of driving frequency.

## STATISTICAL MECHANICS I

Consider a thermally isolated container filled with a monatonic gas which may be treated classically. A small hole is made in the container. allowing some of the gas to escape. The size of the hole is small compared to the mean free path of the gas. The escape rate is slow enough that the remaining gas rapidly reaches equilibrium by collisions among the gas atoms.

a) What is the average kinetic energy per particle of the gas that escapes when the gas inside the container is at temperature $T$?

b) After a certain time. a fraction $N/N_0$ of the initial gas atoms remains in the container. What is the temperature $T$ of the remaining gas in terms of the initial temperature $T_0$?

## STATISTICAL MECHANICS II:

a) Show that for an extreme relativistic noninteracting Fermi or Bose gas $\left(E(\vec{k}) = c\hbar|\vec{k}|\right)$, the relation

$$PV = \frac{1}{3}U$$

holds, where $U$ is the internal energy.

HINT: use the relation $PV = kT \ln Z$ where the grand partition function

$$Z(\mu, T) = \begin{cases} \prod_{\vec{k}}(1 + e^{-(E(\vec{k})-\mu)/kT}) & \text{for fermions} \\ \prod_{\vec{k}}(1 - e^{-(E(\vec{k})-\mu)/kT})^{-1} & \text{for bosons} \end{cases}$$

b) Consider a region of space at very high temperature $T$ so that pair production of electrons and positrons occurs. By considering the process

$$e_+ + e_- \longrightarrow \gamma$$

as a chemical reaction, what is the condition for equilibrium?

For $kT \gg mc^2$, the electrons and positrons may be treated in the extreme relativistic limit $(E(\vec{k}) = c\hbar|\vec{k}|)$. What is the electron energy per unit volume? What is the ratio of the partial pressure of the electrons to that of blackbody radiation? [You may express your answers in terms of dimensionless integrals which need not be evaluated.]

QUANTUM MECHANICS I:

Consider the one-dimensional problem of a light particle (electron) of mass $m$ attracted to two heavy particles (nuclei) of mass $M$ with $\left(\frac{m}{M} \ll 1\right)$ by attractive $\delta$-functions

$$V = -\lambda[\delta(x - R_1) + \delta(x - R_2)]$$

where $x$ is the coordinate of the electron and $R_1$ and $R_2$ are the coordinates of the two "nuclei."

Use the Born-Oppenheimer approximation $(\frac{m}{M} \ll 1)$ to find the interatomic potential $U(R)$ where $R = R_1 - R_2$. Discuss the solution qualitatively showing that $U$ is attractive so that we obtain a "molecular" ion.

QUANTUM MECHANICS II:

a) Use the WKB approximation to obtain a formula for the number of bound states in an arbitrary attractive spherically symmetric potential in three dimensions. Do this as a function of the angular momentum $\ell$.

b) Show that an attractive potential which falls as $1/r^2$ or slower at large distances has an infinite number of bound states for $\ell$ less than a critical $\ell_c$.