General Exam Fall 1997 Solutions

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Mechanics Problem 1

(a)

Assume: Thrust per engine = 56000 pounds, which is approximately 2500 kg.

The speed is 0.6 sound velocity, which is approximately 200m/s.

The power per engine is 5×10^6 kg·m/s = $5 \times 10^6/75$ horse power = $\frac{2}{3} \times 10^6$ hp. Thus the time of flight is $\frac{2}{3} \times 10^6$ horses.// A car of 100 hp takes 4 liters of fuel to run $\frac{1}{2}$ horses. Thus, $\boxed{\text{4 liters of fuel is approximately 50hp } \times horse}$

Fuel for one engine is

$$4 \times (4 \times \frac{2}{3} \times 10^5/50) \approx 2 \times 10^4 \text{ (liters)} \tag{1}$$

Fuel for four engine is $\boxed{8 \times 10^4 \text{ liter}}\pm 100\%$

(b)

Fuel of $\frac{1}{2}$ takeoff weight is 180×10^3 kg, which can make 747 fly for 8000 miles.

the distance from New York to London is about 2000 miles. so the fuel needed is:

$$\boxed{4 \times 10^4 \text{kg} \approx 5 \times 10^4 \text{liters}} \pm 100\% \tag{2}$$

Mechanics Problem 2

(a)

Define the effective mass

$$m^* = \frac{T*2}{v^2} \tag{3}$$

Where T is the kinetic energy and v is the velocity.

For object 1:

$$I = \frac{1}{2}a^2 m \tag{4}$$

$$\omega = \frac{v}{a} \tag{5}$$

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \tag{6}$$

$$= \frac{1}{2}(1+\frac{1}{3})mv^2 \tag{7}$$

$$\boxed{m^* = \frac{3}{2}m} \tag{8}$$

For object 2:

$$I = a^2 m \tag{9}$$

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \tag{10}$$

$$\boxed{m^* = 2m} \tag{11}$$

Similarly for object 3:

$$I = \frac{2}{5}a^2 m \tag{12}$$

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \tag{13}$$

$$\boxed{m^* = \frac{7}{5}m} \tag{14}$$

and for object 4:

$$\boxed{m^* = 2m} \tag{15}$$

assuming $a \ll h, l$, the object with smaller effective mass would finish faster. However object 2 and 4 has the same effective mass. In that case it depends on the initial condition of object 4. Object 4 would finish faster if the mass M is on the top of the ring initially. So the order of finish, from fast to slow, is:
if mass M at the bottom:

$$\boxed{3 \to 1 \to 2 \to 4}$$

if mass M on the top:

$$\boxed{3 \to 1 \to 4 \to 2}$$

(b)

The angular momentum of the disc:

$$L = I\omega = I\frac{v}{a}$$

$$\left|\frac{d}{dt}\vec{L}\right| = \frac{2\pi L\sin(\alpha)}{2\pi\tilde{R}/v} = \frac{Lv\sin(\alpha)}{R-\cos(\alpha)a} \tag{16}$$

$$= aF_2\cos(\alpha) - aF_1\sin(\alpha) \tag{17}$$

$$= aMg\cos(\alpha) - aF_1\sin(\alpha) \tag{18}$$

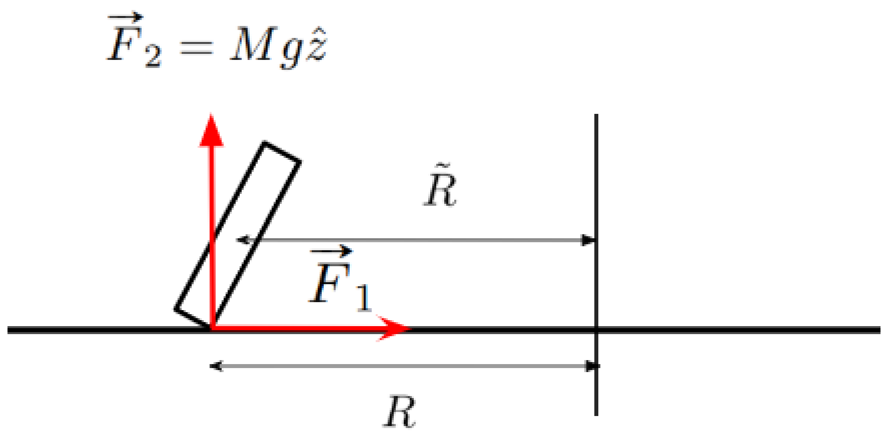

Figure 1: Classical mechanics 2(b)

Thus we get

$$\boxed{I\frac{v}{a}\frac{v\sin(\alpha)}{R-\cos(\alpha)a} = aMg\cos(\alpha) - aF_1\sin(\alpha)} \tag{19}$$

Also notice that

$$\boxed{F_1 = M\frac{v^2}{R-\cos(\alpha)a}} \tag{20}$$

Combine the two boxed equations above, we get

$$\frac{1}{2}Mv^2\frac{a\sin(\alpha)}{\tilde{R}} = aMg\cos(\alpha) - a\frac{Mv^2}{\tilde{R}}\sin(\alpha) \tag{21}$$

Thus, $\boxed{R = \frac{v^2}{g}\frac{(1+\frac{1}{2})\sin(\alpha)}{\cos(\alpha)} + a\cos(\alpha)}$

(c)

Assume that the friction $\boxed{\vec{F} = -a\vec{v}}$ acts on the center of mass. Applying Newton's second law to both rotational and tangential motions:

$$\frac{d}{dt}L = -\frac{I}{a}\frac{d}{dt}v \tag{22}$$

$$M\frac{d}{dt}v = -\frac{I}{a^2}\frac{d}{dt}v - av \tag{23}$$

We get a first order ODE that we can solve for $v(t)$

$$\boxed{\frac{d}{dt}v = -\frac{av}{M+\frac{I}{a^2}}} \tag{24}$$

Consider the rate of energy loss:

$$\frac{d}{dt}(T+V) = -av^2 \tag{25}$$

$$= \frac{d}{dt}(\frac{1}{2}(M+\frac{I}{a^2})v^2 + a\cos(\alpha)Mg) \tag{26}$$

$$= (M+\frac{I}{a^2})v(-\frac{av}{M+\frac{I}{a^2}}) + aMg(-\sin(\alpha))\dot{\alpha} \tag{27}$$

$$= -av^2 - aMg\sin(\alpha)\dot{\alpha} \tag{28}$$

Thus, $\boxed{\dot{\alpha} = 0}$, which means we can solve eqn 24 and plug it into the answer from the previous section.

$$\boxed{v(t) = v_0 e^{-\frac{at}{M+\frac{I}{a^2}}}} \tag{29}$$

$$\boxed{R(t) = \frac{v(t)^2}{g} \frac{\frac{3}{2}\sin(\alpha)}{\cos(\alpha)} + a\cos(\alpha)} \tag{30}$$

Electromagnetism Problem 1

(a)

The emf induced in one loop is:

$$\mathcal{E} = -\frac{1}{c}\frac{\partial \phi}{\partial t} = \frac{i\omega}{c}AB = \frac{i\omega}{c}EA \tag{31}$$

Impedance of a loop:

$$Z = R + i(\omega L - \frac{1}{\omega C}) \tag{32}$$

The current running in a loop is then

$$I = \frac{\mathcal{E}}{Z} = \frac{i\omega EA}{c(R + i(\omega L - \frac{1}{\omega C}))} \tag{33}$$

The dissipated power:

$$P = \langle \bar{I}\mathcal{E}\rangle_{\text{average over cycle}} = \frac{\frac{1}{2}(\omega EA/c)^2 R}{R^2 + (\omega L - \frac{1}{\omega C})^2} \tag{34}$$

Power balance:

$$\frac{dS}{dz} = -nP = -\frac{S}{a} \tag{35}$$

where $n = c^{-3}$ is the concentration of the loops. The temporal average of Poynting vectors is $\langle S\rangle = \frac{c}{8\pi}E^2$. And then $a = \frac{(R^2+(\omega L - \frac{1}{\omega C})^2)c^3}{4\pi\omega^2 A^2 R}$ is the length of power attenuation: $S(z) = S(0)e^{-\frac{z}{a}}$. Angular distribution of radiated power is that of a magnetic dipole radiation. The dipole is oriented along the $\hat{y}$ axis, and thus $\frac{dP}{d\Omega} \sim \sin^2(\theta)$, where θ is the polat angle measured from $\hat{y}$ axis.

$$\boxed{\frac{dP}{d\Omega} = \frac{3P_{rad}}{8\pi}\sin^2(\theta)} \tag{36}$$

where $P_{rad} = \frac{c}{3}k^4 I^2 A^2$.
Check:

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin^3(\theta) = 2\pi \int_{-1}^{1} dx(1-x^2) \tag{37}$$

$$= 4\pi\frac{2}{3} = \frac{8\pi}{3} \tag{38}$$

(b)

For a loop with a unit vector $\vec{n}_0$ normal to its plane, angular distribution of radiated power is:

$$\frac{dP}{d\Omega}(\vec{n}) \sim (1 - (\vec{n}\cdot\vec{n}_0)^2)(\vec{n}_0 \cdot \vec{n}_B)^2 \tag{39}$$

$\vec{n}$ points to the observer and $\vec{n}_B$ is the unit vector along the magnetic field $\vec{B}$. Write the equation above in vector components:

$$\frac{dP}{d\Omega}(\vec{n}) \sim (1 - (n^x n_0^x + n^y n_0^y + n^z n_0^z)^2)(n_0^y)^2 \tag{40}$$

Averaging over orientations of $\vec{n}_0$:

$$\langle (n_0^y)^2\rangle = \frac{1}{3} \tag{41}$$

$$\langle (n_0^y)^4\rangle = \frac{1}{2}\int_0^1 (1-x^2)^2 dx = \frac{4}{15} \tag{42}$$

$$\langle (n_0^x)^2(n_0^y)^2\rangle = \frac{1}{4}\int_0^1 (1-x^2)x^2 dx = \frac{1}{30} \tag{43}$$

$$\langle (n_0^z)^2(n_0^y)^2\rangle = \frac{1}{4}\int_0^1 (1-x^2)x^2 dx = \frac{1}{30} \tag{44}$$

$$\langle \frac{dP}{d\Omega}(\vec{n})\rangle_{n_0} \sim \frac{1}{3}\frac{4}{15}(n^y)^2 - \frac{1}{30}((n^x)^2(n^z)^2) = \frac{9}{30} - \frac{7}{30}(n^y)^2 \tag{45}$$

Thus $\langle \frac{dP}{d\Omega}(\vec{n})\rangle \sim 2 + 7((n^x)^2(n^z)^2) = \boxed{2 + 7\sin^2(\theta)}$

(c)

For unpolarized wave, $\vec{B}$ is randomly oriented withn x-y planem Thus from part (b),

$$\langle \frac{dP}{d\Omega}(n) \rangle_{n_0,B} = \langle 9 - 7(\vec{n} \cdot \vec{n_B})^2 \rangle_B = 9 - \frac{7}{2}((n^x)^2 + (n^y)^2) = \boxed{\frac{11}{2} + \frac{7}{2}(n^z)^2} \tag{46}$$

Or: $\boxed{\langle \frac{dP}{d\Omega}(n) \rangle_{n_0,B} \sim 11 + 7\cos^2(\theta^z)}$

(d)

By uncertainty principle, the angle of divergence is $\delta\theta = \frac{\lambda}{r_0}$

(e)

This part can be interpreted in several different ways. All solutions are accepted as correct. This part, however, was not graded.

(f)

Magnetic moment of one loop:

$$\vec{m} = \frac{1}{c} I A \hat{y} = \frac{\mathcal{E} A}{Z} \hat{y} = \frac{i\omega A^2}{cZ} \vec{B} \tag{47}$$

Thus effective magnetic permeability is:

$$\mu_{eff} = 1 + 4\pi n \frac{i\omega A^2}{c^2 Z(\omega)} \tag{48}$$

From wave equation in effective medium:

$$k^2 = \frac{\omega^2}{c^2} \epsilon_{eff} \mu_{eff} \Rightarrow k = \frac{\omega}{c}(1 + 2\pi n \frac{i\omega A^2}{c^2 Z(\omega)}) \tag{49}$$

Thus the attenuation length of a is given by

$$\frac{1}{a} = 2Im(k) = 4\pi n \frac{\omega^2 A^2 R}{c^3 (R^2 + (\omega L - \frac{1}{\omega C})^2)} \tag{50}$$

Thus, $\boxed{a = \frac{c^3}{4\pi n} \frac{R^2 + (\omega L - \frac{1}{\omega C})^2}{\omega^2 A^2 R}}$

Electromagnetism Problem 2

(a)

Current in the antenna:

$$\begin{cases} I(z) \sim cos(kz+\phi)e^{i\omega t} \Rightarrow I(z) = I\sin\left(k(\frac{l}{2}-|z|)\right) \\ I(z=\pm\frac{l}{2}) = 0 \\ k = \frac{\omega}{c} \end{cases} \tag{51}$$

$$A(r) = \frac{\sin(\theta)}{rc}\int_{-\frac{l}{2}}^{\frac{l}{2}} I(z)e^{ikz\cos(\theta)}dz \tag{52}$$

$$= \frac{2\sin(\theta)}{rc}I\int_0^{\frac{l}{2}}\sin\left(k(\frac{l}{2}-z)\right)\cos(kz\cos(\theta))dz \tag{53}$$

$$= \frac{2\sin(\theta)}{rc}I\int_0^{\frac{l}{2}}\frac{1}{2}[\sin\left(k(\frac{l}{2}-z(1-\cos(\theta)))\right)+\sin\left(k(\frac{l}{2}-z(1+\cos(\theta)))\right)] \tag{54}$$

$$= \frac{I\sin(\theta)}{rc}[\frac{\cos\left(k(\frac{l}{2}-z(1-\cos(\theta)))\right)}{k(1-\cos(\theta))}+\frac{\cos\left(k(\frac{l}{2}-z(1+\cos(\theta)))\right)}{k(1+\cos(\theta))}]\Big|_0^{\frac{l}{2}} \tag{55}$$

$$= \frac{2I}{krc\sin(\theta)}(\cos\left(\frac{kl}{2}\cos(\theta)\right)-\cos\left(\frac{kl}{2}\right)) \tag{56}$$

$$\vec{B} = i\vec{k}\times\vec{A} \Rightarrow \langle S\rangle_{\text{average over cycle}} = \langle\frac{c}{4\pi}B^2\rangle = \frac{ck^2}{4\pi}\frac{A^2}{2} \tag{57}$$

Thus,

$$\boxed{\frac{dP}{d\Omega} = \langle S\rangle r^2 = \frac{I^2}{2\pi c\sin^2(\theta)}(\cos\left(\frac{kl}{2}\cos(\theta)\right)-\cos\left(\frac{kl}{2}\right))^2} \tag{58}$$

When $kl \ll 1$, $\cos\left(\frac{kl}{2}\cos(\theta)\right)-\cos\left(\frac{kl}{2}\right) = \frac{1}{2}(\frac{kl}{2})^2(1-\cos^2(\theta)) = \frac{k^2l^2}{8}\sin^2(\theta)$. Thus,

$$\boxed{\frac{dP}{d\Omega} = \frac{I^2k^4k^4\sin^2(\theta)}{2^7\pi c}} \tag{59}$$

Feeding current $I(z=0) = \frac{Ikl}{2} = \omega Q$, $I = \frac{2\omega C}{kl}V = \frac{2c}{l}CV$,where C is the capacitance.

(b)

When $kl \gg 1$, radiated power oscillates as a function of θ (as shown in the figure)

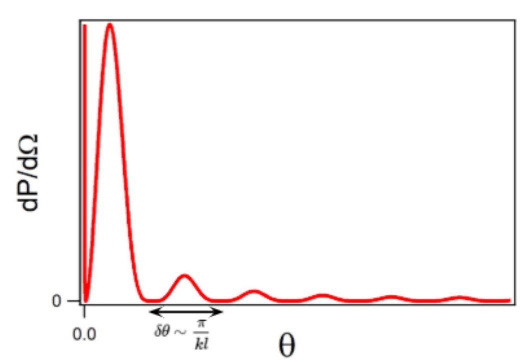

Figure 2: Electricity and Magnetism 2(b)

Averaging over the fast oscillation gives:

$$\langle \frac{dP}{d\Omega} \rangle \sim \frac{I^2}{\sin^2(\theta)} \tag{60}$$

Feeding current $I(z=0) = I \sin\left(\frac{kl}{2}\right) = \frac{V}{Z}$, where Z is the impedance.

(c)

sketch of the polar angle of the power radiation maximum as a function of driving frequency

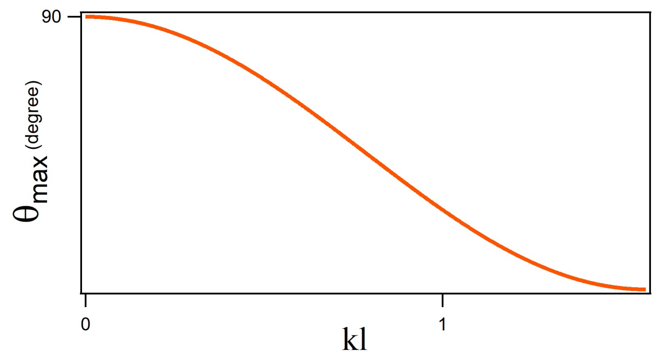

Figure 3: Electricity and Magnetism 2(c)

Statistical Mechanics Problem 1

(a)

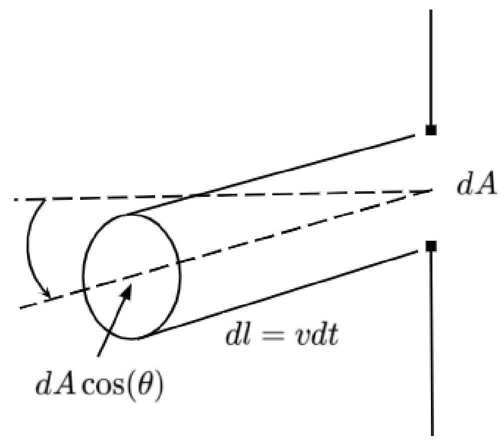

Figure 4: Statistical Mechanics 1

Consider the monatomic gas in a infinitesimal cylinder as shown in the figure.

$$v \equiv |\vec{v}| \tag{61}$$

$$dN = -n(dA\cos\theta)(vdt) = -nv\cos\theta dAdt \tag{62}$$

$$dE = -n(\frac{1}{2}mv^2)(dAcos\theta)(vdt) = -\frac{1}{2}mnv^3\cos\theta dAdt \tag{63}$$

Find $\frac{\langle dE/dt\rangle}{\langle dN/dt\rangle} = \frac{1}{2}m\frac{\langle v\rangle}{\langle v^3\rangle}$. θ and ϕ averages are the same in the numerator and denominator. To do this apply Maxwell's speed distribution $\boxed{p(v) \propto \frac{v^2}{\sigma^3}e^{-\frac{v^2}{2\sigma^2}}}$, where $v \geq 0$. But before that, let's spend some time to derive it. From canonical ensemble with $\langle v_x^2\rangle = \sqrt{\frac{kT}{m}} \equiv \sigma^2$:

$$p(v_x, v_y, v_z) = (\frac{1}{\sqrt{2\pi\langle v_x^2\rangle}})^3 e^{-\frac{v_x^2+v_y^2+v_z^2}{2\langle v_x^2\rangle}} \tag{64}$$

Convert the expression into spherical coordinate using $dv_x dv_y dv_z = v^2\sin\theta d\theta d\phi$. Integration over the solid angle gives 4π as there's no angular dependence.

$$p(v) \equiv \text{probability of } v_x^2 + v_y^2 + v_z^2 \leq v^2 \tag{65}$$

$$= (\frac{1}{\sqrt{2\pi\langle v_x^2\rangle}})^3 \int_0^v e^{-\frac{\xi^2}{2\langle v_x^2\rangle}} 4\pi\xi^2 d\xi \tag{66}$$

Now we are ready to continue.

$$p(v) = \frac{d}{dv}p(v) = (\frac{1}{\sqrt{2\pi\langle v_x^2\rangle}})^3 4\pi v^2 e^{-\frac{v^2}{2\langle v_x^2\rangle}} \tag{67}$$

$$= \frac{2}{\sqrt{2\pi}}\frac{v^2}{\sigma^3}e^{-\frac{v^2}{2\sigma^2}}, v > 0, \text{q.e.d} \tag{68}$$

$$\frac{\langle v^3\rangle}{\langle v\rangle} = \frac{\int_0^\infty v^5 e^{-\frac{v^2}{2\sigma^2}}dv}{\int_0^\infty v^3 e^{-\frac{v^2}{2\sigma^2}}dv} \tag{69}$$

$$\int_0^\infty v^{2n+1}e^{-\frac{v^2}{2\sigma^2}}dv = 2\sigma^2(2\sigma^2)^n n! \tag{70}$$

Thus, $\frac{\langle v^3\rangle}{\langle v\rangle} = 4\sigma^2$, now use $\sigma = \sqrt{\frac{kT}{m}}$, $\boxed{\frac{\langle dE/dt\rangle}{\langle dN/dt\rangle} = 2m\sigma^2 = 2kT}$

(b)

$$E = \frac{3}{2}NkT \tag{71}$$

$$dE = \frac{3}{2}NkdT + \frac{3}{2}kTdN = 2kTdN \tag{72}$$

$$\Rightarrow \frac{dT}{T} = \frac{1}{3}\frac{dN}{N} \Rightarrow \ln\frac{T}{T_0} = \frac{1}{3}\ln\frac{N}{N_0} \tag{73}$$

$$\boxed{\frac{T}{T_0} = (\frac{N}{N_0})^{\frac{1}{3}}} \tag{74}$$

Statistical Mechanics Problem 2

(a)

For relativistic particles, the dispersion is $\epsilon(k) = c\hbar k$. First calculate the density of states (DOS) for relativistic particles. The number of states in a sphere with radius k in momentum space is:

$$\#(k) = 2(\frac{4}{3}\pi k^3)\frac{V}{(2\pi)^3} \tag{75}$$

Differentiate with respect to k :

$$\frac{d\#}{dk} = \frac{1}{\pi^2}k^2 V \tag{76}$$

and write in terms of energy ϵ gives the DOS:

$$\frac{d\#}{d\epsilon} \equiv D(\epsilon) = (\frac{1}{c\hbar})^3 V\epsilon^2 \tag{77}$$

Recall the statistic for quantum gas:

$$\bar{n} = (e^{\frac{\epsilon-\mu}{kT}} \pm 1)^{-1} \tag{78}$$

where $+$ is taken for fermions and $-$ is taken for bosons. The internal energy is then

$$U = \sum_{\text{states}} \epsilon(\vec{k})\bar{n}(\epsilon) = \int_0^\infty \epsilon D(\epsilon)\bar{n(\epsilon)}d\epsilon \tag{79}$$

$$= \frac{1}{\pi^2}(\frac{1}{c\hbar})^3 V \int_0^\infty \epsilon^3(e^{\frac{\epsilon-\mu}{kT}} \pm 1)^{-1}d\epsilon \tag{80}$$

Using the grand partition function given in the question to calculate PV:

$$PV = kT\ln Z = kT\sum_{\text{states}} \ln\left(1 \pm e^{-(\epsilon-\mu)/kT}\right)^{\pm 1} \tag{81}$$

$$= \pm kT\int_0^\infty D(\epsilon)\ln\left(1 \pm e^{-(\epsilon-\mu)/kT}\right)d\epsilon \tag{82}$$

$$= \pm[kT(\frac{1}{\pi})^2(\frac{1}{c\hbar})^3 V]\int_0^\infty \epsilon^2\ln\left(1 \pm e^{-(\epsilon-\mu)/kT}\right)d\epsilon \Rightarrow \text{integral by parts} \tag{83}$$

$$= (-)(\pm)(\pm)[kT(\frac{1}{\pi})^2(\frac{1}{c\hbar})^3 V]\int_0^\infty \frac{1}{3}\epsilon^3(1 \pm e^{-(\epsilon-\mu)/kT})^{-1}e^{-\frac{\epsilon-\mu}{kT}}(-\frac{1}{kT})d\epsilon \tag{84}$$

$$= \frac{1}{3}\frac{1}{kT}[kT(\frac{1}{\pi})^2(\frac{1}{c\hbar})^3 V]\int_0^\infty \epsilon^3(e^{(\epsilon-\mu)/kT} \pm 1)^{-1}d\epsilon \tag{85}$$

$$= \boxed{\frac{1}{3}U} \tag{86}$$

(b)

For an equilibrium reaction, $\sum_i \partial_i\mu_i = 0$, Note that the chemical potential for photon is $\mu_\gamma = 0$. Thus:

$$(1)\mu_{e_-} + (1)\mu_{e_+} + (1)0 = 0 \tag{87}$$

So $\mu_{e_-} + \mu_{e_+} = 0$, but $\mu_{\text{fermion}} \geq 0$, this implies that $\boxed{\mu_{e_-} = \mu_{e_+} = 0}$. From the result in (a) with $\mu_{e_-} = 0$, one has:

$$\boxed{\frac{U_{e_-}}{V} = \frac{1}{\pi^2}(\frac{1}{c\hbar})^3\int_0^\infty \epsilon^3\frac{1}{e^{\frac{\epsilon}{kT}} + 1}d\epsilon} \tag{88}$$

For photon, there is also a factor of 2 degeneracy due, this time, to polarization.

$$\boxed{\frac{U_\gamma}{V} = \frac{1}{\pi^2}(\frac{1}{c\hbar})^3\int_0^\infty \epsilon^3\frac{1}{e^{\frac{\epsilon}{kT}} - 1}d\epsilon} \tag{89}$$

And since $P = \frac{1}{3}\frac{U}{V}$ in each case, $\boxed{\frac{P_{e_-}}{P_\gamma} = \frac{\int_0^\infty \epsilon^3(e^{\frac{\epsilon}{kT}} + 1)^{-1}d\epsilon}{\int_0^\infty \epsilon^3(e^{\frac{\epsilon}{kT}} - 1)^{-1}d\epsilon}}$

Quantum Mechanics Problem 1

For convenience, lets set the origin of our coordinate at the center of two delta-wells and relabel the positions of the delta-wells to be $\pm L$ (as shown in the figure). Thus $L = \frac{R_1 - R_2}{2} = \frac{R}{2}$.

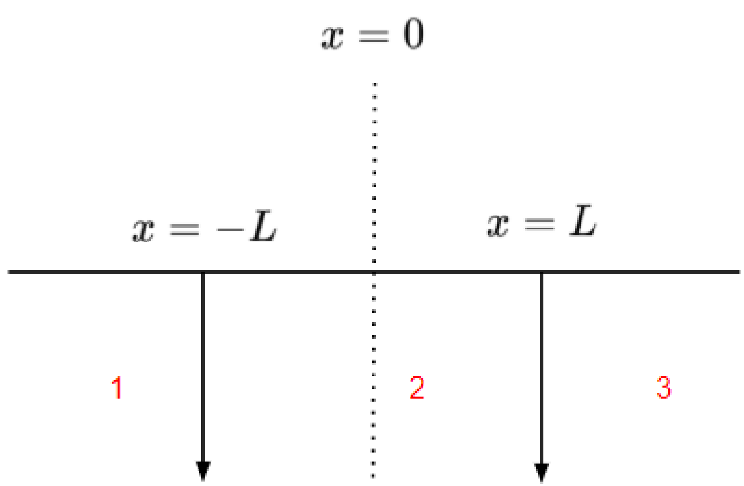

Figure 5: quantum mechanics 1

Solve the Schrodinger equation with the double delta-function potential:

$$[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \lambda\delta(x-L) - \lambda\delta(x+L)]\psi = E\psi \tag{90}$$

The solution to the equation is the same as free particles except for the discontinuities in first derivative at the delta functions. Take $x = L$ as an example: Integrate from the left side of the delta function $L - \epsilon$ to the right side $L + \epsilon$

$$-\frac{\hbar^2}{2m}\{\frac{d\psi}{dx}\bigg|_{L+\epsilon} - \frac{d\psi}{dx}\bigg|_{L-\epsilon}\} - \lambda\psi(L) = 0 \tag{91}$$

$$\frac{d\psi}{dx}\bigg|_{L+\epsilon} - \frac{d\psi}{dx}\bigg|_{L-\epsilon} = -\frac{2m\lambda}{\hbar^2}\psi(L) \tag{92}$$

This gives us the continuity condition at the delta function. The condition is the same at $x = -L$. The solutions in three regions:

$$\psi(x) = e^{-\alpha x} \text{ for } x > L \tag{93}$$

$$\psi(x) = A[e^{\alpha x} + e^{-\alpha x}] \text{ for } -L < x < L \tag{94}$$

$$\psi(x) = e^{\alpha x} \text{ for } x < -L \tag{95}$$

where α is a position number which satisfies $E = -\frac{\hbar^2}{2m}\alpha^2$. Other redundant undetermined coefficients are eliminated by the constraint that the the solution should an even function. To determine the last unknown coefficient, match the boundary condition at the delta functions to make the wave function continuous. Apply the boundary conditions at L:

$$-\alpha e^{-\alpha L} - \{A\alpha e^{\alpha L} - A\alpha e^{-\alpha L}\} = -\frac{2m\lambda}{\hbar^2}e^{-\alpha L} \tag{96}$$

$$e^{-\alpha L} = A[e^{\alpha L} + e^{-\alpha L}] \tag{97}$$

Combine these two equations gives:

$$\alpha\{1 + \frac{e^{2\alpha L} - 1}{e^{2\alpha L} + 1}\} = \frac{2m\lambda}{\hbar^2} \tag{98}$$

$$2e^{2\alpha L} = \frac{2m\lambda}{\hbar^2\alpha}(e^{2\alpha L} + 1) \tag{99}$$

$$\alpha = \frac{m\lambda}{\hbar^2}(1 + e^{-2\alpha L}) \tag{100}$$

Writing α in terms of E and L instead of R gives what it asks for in the question. This is a transcendental equation and need to be solved graphically:

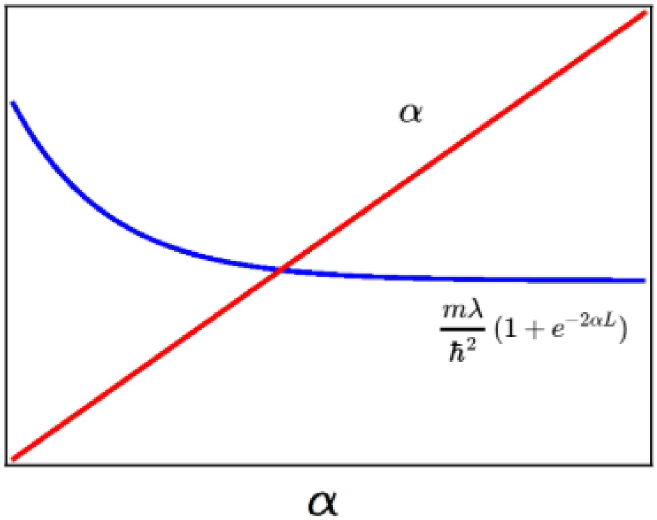

Figure 6: solution to the transcendental equation

Notice that α approaches $\frac{m\lambda}{\hbar^2}$ in large L limit ,which is the bound state energy for single Delta function potential. This means that the energy gets more positive when L is large in single-atom limit. Thus, as $E_{\text{molecule}} < E_{\text{atom}}$, the potential is thus attractive.

Quantum Mechanics Problem 2

(a)

For a spherically symmetric central potential, we can write it into an effective potential:

$$V(r) \to V(r) + \frac{l(l+1)\hbar^2}{2mr^2} = V_{eff}(r) \tag{101}$$

WKB quantization:

$$\frac{1}{\hbar}\int_{r_{min}}^{r_{max}} (2m(E - V_{eff}(r')))^{\frac{1}{2}} dr' = (n + \frac{1}{2})\pi \tag{102}$$

For a given $V(r)$, find the the maximum E which gives (classically) bound motion in the potential $V(r)$. Find the associated r_{min} and r_{max} . Compute n with the above formula. For l, there are $2l+1$ states with each energy. So, you need to multiply by $2l+1$.

(b)

For large r, suppose $V(r) = -\frac{k}{r^2}$. $V_{eff} = -\frac{k}{r^2} + \frac{l(l+1)\hbar^2}{2mr^2}$, where k is a positive constant. Then, bound states only exists when $V_{eff} < 0$, which means $\frac{l(l+1)}{2m} < k$. Thus the critical value is $l_c = \frac{\sqrt{8mk+1}-1}{2}$. The following plot shows the condition for bound states to exist:

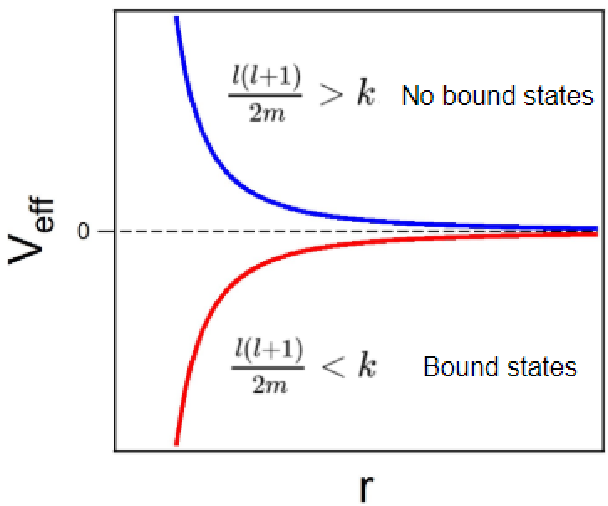

Figure 7: Conditions for bound states to exist

The max energy E_{max} when the bound state exists is 0 at $r_{max} = \infty$.

$$\int_0^\infty \sqrt{k - \frac{l(l+1)\hbar^2}{2m}}\frac{dr'}{r'} = \infty \tag{103}$$

The number of bound states is infinity. For slower potentials, the result is the same. suppose $V(r) \sim -\frac{k}{r^{2-\epsilon}}$

$$\int_0^\infty \sqrt{k - \frac{l(l+1)\hbar^2}{2m}}\frac{dr'}{r'^{1-\frac{\epsilon}{2}}} = \infty \tag{104}$$