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DEPARTMENT OF PHYSICS

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DOCTORAL GENERAL EXAMINATION

PART II

SEPTEMBER 11, 1998

FIVE HOURS

1. This examination is divided into four sections, Mechanics, Electricity & Magnetism, Statistical Mechanics, and Quantum Mechanics, with two problems in each. It is advisable to carefully read both problems in each section before making your choice. Submit ONLY one problem per section. IF YOU SUBMIT MORE THAN ONE PROBLEM FROM A SECTION, BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.
2. Use a separate fold of paper for each problem, and write your name on each fold. Include the problem number with each solution.
3. Calculators may be used.
4. **No Books or Reference Materials May Be Used.**

## Problem 1 (Mechanics)

A uniform string of length  $L$  is tied at its two ends along the  $x$  axis and stretches from  $x = 0$  to  $x = L$ . The mass per unit length of the string is  $\rho$ . For times  $t < 0$  the tension is a constant,  $T_0$ , and the string is given to be oscillating in the form

$$y(x, t) = A \sin(\omega t) \sin\left(\frac{\pi x}{L}\right), \quad t < 0, \quad (1)$$

where  $y$  is the transverse displacement of the string.

a) Write the Lagrangian  $L$  for a string of mass per unit length  $\rho$  and tension  $T_0$ . Note that in addition to kinetic energy from the motion, there is potential energy from the stretching of the string. From this Lagrangian find the Lagrangian equation of motion of the string.

b) Using Newton's second law applied to an infinitesimal piece of string rederive the equation of motion obtained in part (a).

c) What is the value of the frequency  $\omega$  in the expression (1) for  $y(x, t)$  ?

We now consider the possibility that the tension is time dependent. For this purpose one may simply replace  $T_0$  by  $T(t)$  in the Lagrangian and in the equation of motion. In the time interval  $0 < t < t_0$  the tension is increased to a slightly higher value  $T_0(1 + \epsilon)$ , with  $\epsilon$  a small constant, after which the tension returns to the value  $T_0$ .

d) Assuming a simple 'separation of variables' form  $y(x, t) = X(x)Y(t)$  write out the equations for the functions  $X(x)$ ,  $Y(t)$ . Solve for  $X(x)$ .

e) Rewrite the equation for  $Y(t)$  in the form  $\frac{d^2 Y}{dt^2} + \omega^2 Y = \dots$ , where the dots represent the perturbation term linear in  $\epsilon$ . Using first order perturbation theory in the small constant  $\epsilon$ , solve for  $Y(t)$  for times  $t > t_0$ .

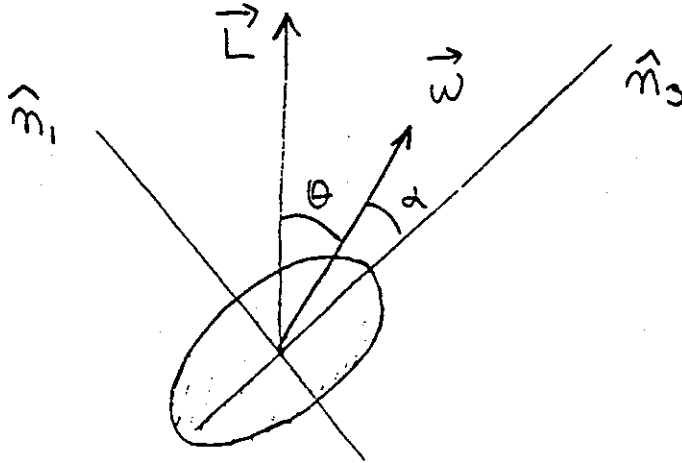
**Useful information** The Green's function for the equation  $\frac{d^2 y}{dt^2} + \omega^2 y = 0$  is

$$G(t, t') = \frac{1}{\omega} \sin[\omega(t - t')], \quad \text{for } t > t' \\ = 0, \quad \text{for } t < t'.$$

It satisfies  $\frac{d^2 G(t, t')}{dt^2} + \omega^2 G(t, t') = \delta(t - t')$ .

## Problem 2 (Mechanics)

Consider a prolate spheroid with no external forces acting on it. At  $t = 0$  the spheroid is given an angular velocity of magnitude  $\omega$  about its center of mass, in a direction which is inclined at angle  $\alpha$  to the axis of symmetry.



Use 'body axes' such that the unit vector  $\hat{n}_3$  points along the symmetry axis of the spheroid, and the unit vectors  $\hat{n}_1$  and  $\hat{n}_2$  lie in the plane perpendicular to this axis. The moments of inertia are  $I_1 = I_2 = I$  and  $I_3 = (1 - a)I$ , where  $0 < a < 1$  is a constant.

- Write the angular velocity vector  $\vec{\omega}$  at  $t = 0$  using the unit vectors  $\hat{n}_1$ ,  $\hat{n}_2$  and  $\hat{n}_3$ . Assume that this vector lies in the  $\hat{n}_1$  -  $\hat{n}_3$  plane at  $t = 0$ .
- Find the angular momentum vector  $\vec{L}$  at  $t = 0$  in this basis.

The *space cone* is generated by the motion of  $\vec{\omega}$  around  $\vec{L}$ , and the *body cone* is generated by the motion of  $\vec{\omega}$  around the symmetry axis of the body. The half angle of the body cone is  $\alpha$  and the half angle of the space cone is  $\theta$  (see figure).

- Find the half angle  $\theta$  of the space cone in terms of  $\alpha$  and  $a$ .
- Find the angular frequency  $\Omega_b$  for the rotation of  $\vec{\omega}$  around the axis of symmetry of the spheroid (as seen by an observer fixed on the spheroid,  $\vec{\omega}$  spins around the symmetry axis).
- Find the angular frequency  $\Omega_p$  for the rotation of  $\vec{\omega}$  around  $\vec{L}$ . Give your answer in terms of  $I$  and the magnitude  $L$  of  $\vec{L}$ .

Useful information:

$$\left(\frac{d\vec{L}}{dt}\right)_s = \left(\frac{d\vec{L}}{dt}\right)_b + \vec{\omega} \times \vec{L}$$

where 's' denotes space fixed coordinates, and 'b' denotes body fixed coordinates.

## Problem 1 (Electromagnetism)

An infinitely long wire on the  $z$ -axis has no current for  $t < 0$  and a constant current  $I$  for  $t > 0$  (in the positive  $z$ -direction). As in an ordinary wire, where the fixed positive ions cancel the charge of the moving electrons, there is no charge density for all  $t$ .

- Express the current density  $\vec{J}(\vec{r}, t)$  using  $\delta$ -functions and/or step functions.
- What is the vector potential  $\vec{A}(\vec{r}, t)$  for this current? Use the retarded Green's functions and evaluate all integrals. Hints:  $\vec{A}(\vec{r}, t) = \vec{A}(\rho, t)$  where  $\rho$  is the normal distance to the wire. As the answer is independent of  $z$ , consider the observer at  $z = 0$ .
- Find the magnetic field  $\vec{B}(\vec{r}, t)$  corresponding to  $\vec{A}(\vec{r}, t)$ . Plot your answer in two ways; in the first show the magnetic field at a fixed point as a function of time, and in the second show the magnetic field at a fixed time as a function of the radial distance to the wire.
- Find the electric field  $\vec{E}(\vec{r}, t)$ . What is its behavior at large times?

### Useful Formulas:

$$\vec{A}(\vec{r}, t) = \frac{1}{c} \int d^3r' \frac{\vec{J}[\vec{r}', t' = t - \frac{|\vec{r} - \vec{r}'|}{c}]}{|\vec{r} - \vec{r}'|}$$

$$\vec{E} = -\nabla\Phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$$

$$\int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u$$

In cylindrical coordinates

$$\nabla \times \vec{a} = \begin{vmatrix} \hat{e}_\rho & \hat{e}_\varphi & \hat{e}_z \\ \frac{\partial}{\partial \rho} & \frac{1}{\rho} \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ a_\rho & a_\varphi & a_z \end{vmatrix}$$

## Problem 2 (Electromagnetism)

A plane wave travelling along the positive  $z$ -direction is incident normally on a uniform material filling the half-space  $z \geq 0$ . The material has a constant conductivity  $\sigma > 0$ , and  $\epsilon = \mu = 1$ .

The incident field (for  $z < 0$ ) is of the form

$$\vec{E}_{\text{inc}}(\vec{r}, t) = \text{Re} \left\{ E_0 e^{i(kz - \omega t)} \hat{e}_x \right\}, \quad (1)$$

with  $k = \frac{\omega}{c}$  and  $E_0$  a real constant. Consider the simple *ansatz* for the wave in the conductor ( $z > 0$ )

$$\vec{E}_{\text{con}}(\vec{r}, t) = \text{Re} \left\{ \vec{E}_c(\vec{r}) e^{-i\omega t} \right\}. \quad (2)$$

Here, the possibly complex vector  $\vec{E}_c(\vec{r})$  is to be determined. For  $z < 0$  we assume that in addition to the incident wave there is a reflected wave of the form

$$\vec{E}_{\text{ref}}(\vec{r}, t) = \text{Re} \left\{ E_r e^{i(-kz - \omega t)} \vec{e}_x \right\}, \quad (3)$$

propagating in the  $(-z)$  direction. Here  $E_r$  is a constant to be found.

- a) Consider Maxwell's equations for the time-independent fields  $\vec{E}_c(\vec{r})$ ,  $\vec{B}_c(\vec{r})$  in the conducting media. Use  $\vec{J}_c = \sigma \vec{E}_c$ , and show that the differential equation for  $\vec{E}_c(\vec{r})$  takes the form

$$\left\{ \nabla^2 + k^2 \left( 1 + i \frac{4\pi\sigma}{\omega} \right) \right\} \vec{E}_c(\vec{r}) = 0. \quad (4)$$

- b) Assume now that the fields in the conductor are of the form

$$\begin{aligned} \vec{E}_c(\vec{r}) &= E_c e^{i\beta z} \hat{e}_x, \\ \vec{B}_c(\vec{r}) &= B_c e^{i\beta z} \hat{e}_y. \end{aligned} \quad (5)$$

What is the value of  $\beta$  in terms of  $k, \sigma, \omega$ ? Find  $B_c$  in terms of  $E_c$ .

- c) For the reflected wave find the time-independent magnetic field in terms of the constant  $E_r$  introduced in (3).

- d) What are the relevant boundary conditions for  $\vec{E}$  and  $\vec{B}$  at the boundary? Set up and solve the system of equations that determine  $E_c$  and  $E_r$  in terms of  $E_0$ ,  $k$  and  $\beta$ .
- e) Using the stress-tensor  $T_{ij}$  find the pressure on the conducting wall due to the radiation. (Draw a sketch showing the surface you will use) Express your answer in terms of  $E_c$ ,  $k$  and  $\beta$ .

**Useful Information:**

$$\nabla \times (\nabla \times \vec{a}) = \nabla(\nabla \cdot \vec{a}) - \nabla^2 \vec{a}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

$$T_{ij} = \frac{1}{4\pi} \left[ E_i E_j + B_i B_j - \frac{1}{2} \delta_{ij} (|\vec{E}|^2 + |\vec{B}|^2) \right]$$

# Statistical Mechanics: Problem #1

1. The *Dieterici equation of state* relates the pressure  $P$  of an interacting gas to its temperature  $T$ , and specific volume  $v = V/N$ , by

$$P = \frac{k_B T}{(v-b)} \exp \left[ -\frac{a}{k_B T v} \right]$$

where  $a$  and  $b$  are positive constants.

- (a) Sketch typical isotherms  $P(v)$  predicted by the above equation for high and low temperatures. What is wrong with this equation? Sketch the isotherms of a real gas at low temperatures.
- (b) Find the coordinates  $(P_c, v_c, T_c)$  of the *critical point* of this gas.
- (c) Find the singular temperature dependence of the isothermal susceptibility

$$K_T \equiv -\frac{1}{v} \frac{\partial v}{\partial P} \Big|_{T,N}$$

on approaching the critical point  $T \rightarrow T_c$ , at  $v = v_c$ .

- (d) Find the pressure  $P(v, T_c)$  on the critical isotherm  $T = T_c$  to lowest non-vanishing order in  $(v - v_c)$ .

## Statistical Mechanics: Problem #2

2. Consider a system of non-interacting spin 1/2 *fermions* at temperature  $T$ , in volume  $V$ , and chemical potential  $\mu$ .

(a) Show that the average occupation numbers of single-particle states of energy  $\mu \pm \Delta$  are related by

$$\langle n(\mu + \Delta) \rangle + \langle n(\mu - \Delta) \rangle = 1$$

where  $\Delta$  is any constant energy.

Now for the rest of this problem, assume that the single particle states come in pairs of positive and negative energies,

$$\epsilon_{\pm}(\mathbf{k}) = \pm \hbar c |\mathbf{k}|$$

independent of spin, and that at  $T = 0$  all negative energy states are occupied and all positive energy states are empty.

(b) What is the chemical potential of this system at  $T = 0$ ? Use the result from (a) to show that  $\mu$  does not change with  $T$ .

(c) Show that the mean total excitation energy of this system at non-zero temperature satisfies

$$E(T) - E(0) = 4V \int_0^{\infty} \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{\epsilon_+(\mathbf{k})}{\exp[\beta \epsilon_+(\mathbf{k})] + 1} \quad \text{where} \quad \beta \equiv \frac{1}{k_B T}$$

(d) Evaluate the heat capacity  $C_V$  of this system, using

$$\int_0^{\infty} dx \frac{x^3}{e^x + 1} = \frac{7\pi^4}{120}$$



## Quantum Mechanics: Problem #1

1. Consider a particle of charge  $q$  and mass  $m$  confined to the  $x$ - $y$  plane and subject to a harmonic oscillator potential  $V = \frac{1}{2}m\omega^2(x^2 + y^2)$  and a uniform electric field of magnitude  $E$  oriented along the positive  $x$ -direction.

- (a) What is the Hamiltonian for the system?
- (b) What are the eigenvalues and associated degeneracies for this Hamiltonian?

Now assume the particle is in the ground state of this system.

- (c) If the electric field is suddenly turned off, show that the probability of finding the particle with energy  $(n + 1)\hbar\omega$  is given by a Poisson distribution

$$P(n) = \frac{e^{-\lambda}\lambda^n}{n!}$$

Find  $\lambda$ .

[You may find the following expressions useful:

$$a_x^+ = \left(\frac{1}{2m\omega\hbar}\right)^{1/2} (m\omega x - ip_x)$$

and  $\exp[A + B] = \exp(A)\exp(B)\exp\left(-\frac{1}{2}[A, B]\right)$  where  $[A, B]$  is a  $c$ -number.]

## Quantum Mechanics: Problem #2

2. A molecule consisting of three fixed identical atoms in an equilateral triangle captures an extra electron. Assume that in this system the electron is described by a Hamiltonian  $H$ . Ignore the spin of the electron and any other electrons already present in the atoms. To obtain the eigenstates of this captured electron we use a simple basis set consisting of one spherically symmetric localized orbital  $|S_i\rangle$  on each atom  $i$  and assume that they are orthonormal (i.e.,  $\langle S_i|S_j\rangle = \delta_{ij}$ ).
- (a) Suppose that  $\langle S_1|H|S_2\rangle = \langle S_2|H|S_3\rangle = \langle S_3|H|S_1\rangle \equiv V$  are the *only* non-zero matrix elements in this basis set, find the energies for the captured electron.  
[Hint: Use the fact that one of the eigenvalues is  $2V$ , which is non-degenerate.]
- (b) Since  $H$  is invariant under rotations by  $\frac{2\pi}{3}$ , construct all simultaneous eigenstates of energy and rotation for the captured electron. What are the rotational eigenvalues for each eigenstate?  
[Hint: The rotation operator  $R\left(\frac{2\pi}{3}\right)$  satisfies  $R^3 = 1$ .]
- (c) Suppose that at time  $t = 0$  the electron is captured completely by atom #1 in the state  $|S_1\rangle$ . What is the probability of finding this electron on atom #1 at a later time  $t$ ? Describe the motion of this electron.

