MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS

EDUCATION OFFICE
ROOM 4-352

PHONE (617)253-4842
FAX (617) 258-8319

DOCTORAL GENERAL EXAMINATION

PART II

September 9, 1999

FIVE HOURS

1. This examination is divided into four sections, Mechanics, Electricity & Magnetism, Statistical Mechanics, and Quantum Mechanics, with two problems in each. It is advisable to carefully read both problems in each section before making your choice. Submit ONLY one problem per section. IF YOU SUBMIT MORE THAN ONE PROBLEM FROM A SECTION, BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.

2. Use a separate fold of paper for each problem, and write your name on each fold. Include the problem number with each solution.

3. Calculators may be used.

4. **No Books or Reference Materials May Be Used.**

Mechanics *Problem 1*

A particle of mass m moves in one dimension. Its position is given by the coordinate x. When $0 < x < L$, no force acts on it. When $x < 0$ a force $+F$ ($F > 0$, independent of x) acts, and when $x > L$ a force $-F$ acts. The Hamiltonian is

$$H(x,p) = \frac{p^2}{2m} + V(x) .$$

(a) Sketch a graph of $V(x)$.

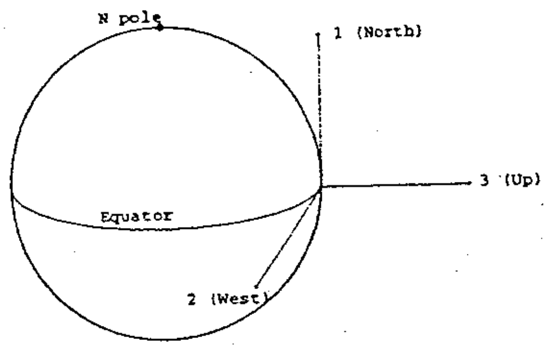
(b) The motion is periodic. When the total energy is E, sketch the trajectory in phase space (i.e., x, p space) for one period, marking the direction of motion along the trajectory.

(c) The limit $F \to \infty$ corresponds to the particle making elastic collisions with fixed walls at $x = 0, L$. Sketch the phase-space trajectory in this limit and calculate the area A of phase space enclosed by it as a function of E.

(d) Now suppose L slowly increases with time, $L(t) = L_0 + ut$, with u much less than the speed of the particle. The limit $F \to \infty$ now corresponds to elastic collisions with one fixed wall and one moving wall. In this limit calculate the change in the speed of the particle after one collision with each wall. Hence show that in the limit $u \to 0$ the quantity $A(E)$ of part (c) does not change with time, even over times of order L_0/u (i.e., A is an adiabatic invariant).

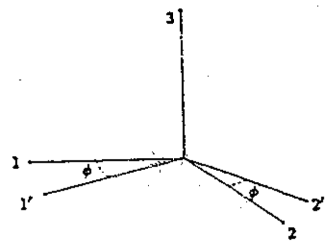
Mechanics *Problem 2*

An idealized gyrocompass consists of a disk spinning about an axis through its center perpendicular to the disk. The axis is mounted so that the center of the disk is fixed and the axis is constrained to be horizontal, but can move freely in the horizontal plane, so that the only torque acting on the disk due to the constraints is about the horizontal direction perpendicular to the axis of the disk. The whole system is fixed at a point on the earth's equator. Define a set of axes rotating with the earth as shown:



so that 3 is vertical, 1 is horizontal pointing north, and 2 is horizontal pointing west.

Let the axis of the disk lie in the (1,2) plane at angle ϕ to 1. Define a set of axes $1'$, $2'$, 3 as shown.



Treating $\phi(t)$ as if it were known, write an explicit expression for the angular velocity of the $1'2'3$ axes in the coordinate system of your choice, including the contribution due to the rotation of the earth. The angular velocity of the disk is the same as the angular velocity of the axes, plus a spin about $1'$. Let the total angular velocity of the disk have component S about $1'$. Let the moment of inertia of the disk about $1'$ be I and about any axis in the plane of the disk (i.e., in the $2'$, 3 plane) be $\frac{1}{2}I$. Write an explicit expression for the angular momentum of the disk and calculate its time derivative. Use the fact that the torque is about the $2'$ direction to show that S is constant and that $\phi = 0$ or π are the only values for which ϕ can be constant. Calculate the frequency of small oscillations about whichever of these values is stable. (Assume S is much greater than the earth's spin.) Estimate what the spin about the axis should be (in rps) to give an oscillation period of 10 seconds.

Electricity and Magnetism *Problem 1*

(a) A simple oscillator consists of a capacitor C and an inductor L (Figure i). C con-

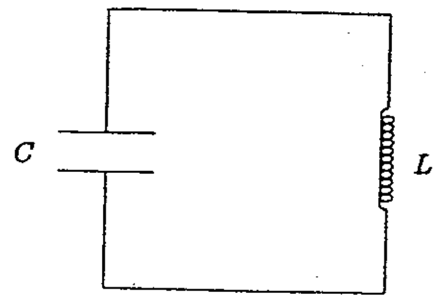


Figure i

sists of two parallel plates of area A_C, distance apart d_C. L is a coil of a total of N turns wrapped around a cylinder of cross-sectional area A_L and height d_L. The period of oscillation is T_0. Find an expression for the speed of light in terms of A_C, d_C, A_L, d_L, N, T_0. Neglect edge effects, end effects, and all effects of material between the plates and inside the coil.

(b) When the capacitor C of part (a) is in series with a resistor R_1 (Figure ii), the de-

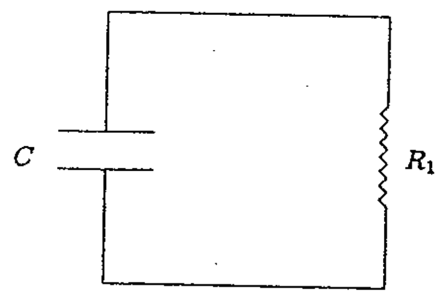


Figure ii

cay of an initial charge is given by

$$Q(t) = Q(0)\exp\left(-\frac{t}{T_1}\right) .$$

When the inductor L of part (a) is in series with a resistor R_2 (Figure iii) the de-

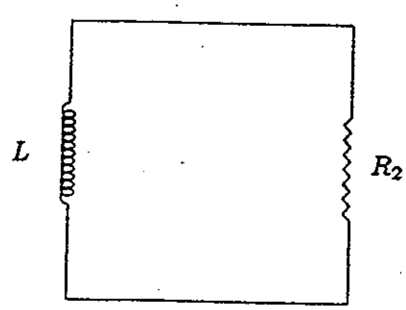


Figure iii

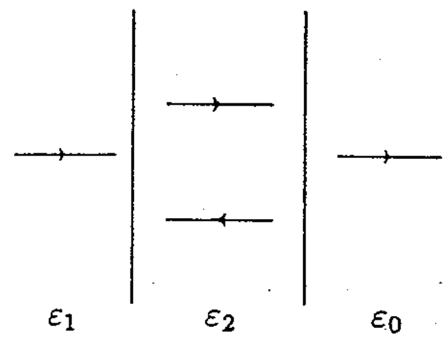
cay of an initial current is given by

$$I(t) = I(0) \exp\left(-\frac{t}{T_2}\right) .$$

Express R_2/R_1 in terms of T_0, T_1, T_2.

Electricity and Magnetism *Problem 2*

(a) Write down a solution of Maxwell's equations representing a plane wave traveling in the $+z$ direction [i.e., with space-time dependence $\exp i(kz - \omega t)$] with the electric field in the x direction and the magnetic field in the y direction, in a dielectric with $\mathbf{D} = \epsilon\mathbf{E}$. What changes are needed for a wave traveling in the $-z$ direction?



(b) A wave as in (a) is incident normally from a medium with $\mathbf{D} = \varepsilon_1\mathbf{E}$ into a layer with $\mathbf{D} = \varepsilon_2\mathbf{E}$ and is then transmitted into vacuum (with $\mathbf{D} = \varepsilon_0\mathbf{E}$). Show that it is possible to have no wave reflected into medium 1 by choosing a particular ε_2 and then making the thickness of the layer 2 exactly one-quarter wavelength (at the given frequency ω).

Statistical Mechanics Problem 1
Fermi and Bose Ideal Gases

Consider an ideal gas of identical Bose particles, confined to a cubical box of side L. Suppose that the energy E of a single particle is given in terms of its momentum $\vec{q}$ by some unspecified function

$$E = E(|\vec{q}|) .$$

Suppose further that each particle has two possible spin states, like the photon.

(a) Write an expression for the free energy $F(L,T) \equiv -kT \ln Z$ of this gas, where T is the temperature, k is the Boltzmann constant, and

$$Z \equiv \sum_{\text{all states}} e^{-E_{\text{tot}}/kT} .$$

The sum is over all states involving any number of particles, and E_{tot} is the total energy of each such state. There is no chemical potential factor, as the chemical potential is taken to be zero. All interactions are to be neglected, so

$$E_{\text{tot}} = \sum_i E(|\vec{q}_i|) ,$$

where $\vec{q}_i$ is the momentum of the i'th particle in the state. Assume for simplicity that the box has periodic boundary conditions, so each face on the boundary is identified with the opposite face. Assume further that the sum over momentum states can be well-approximated by an integral. Your answer should have the form

$$F(L,T) = \int d^3q\, f(|\vec{q}|) ,$$

where $f(|\vec{q}|)$ is an explicit function that you must determine. You cannot carry out the integration over $\vec{q}$, since $E(|\vec{q}|)$ has not been specified, but you should carry out any other sums that appear in your expression.

(b) The entropy of the system is given by

$$S = -\frac{\partial F}{\partial T} .$$

Using the above equation and the definition of F given in part (a), show that the expectation value $\mathcal{E}$ of the total energy of the system can be written as

$$\mathcal{E} = F + TS .$$

— *PROBLEM CONTINUES ON NEXT PAGE* —

(c) Now consider the case in which the particles are photons, so that $E(|\vec{q}|) = c|\vec{q}|$. The integral in the expression for $F(L,T)$ can then be evaluated, giving

$$F(L,T) = -\frac{\pi^2}{45}\frac{(kT)^4}{(\hbar c)^3} L^3 .$$

Suppose that the photon gas, initially at temperature T, is compressed. The length of each side is decreased from L to αL, where $0 < \alpha < 1$. If the compression is very slow, and the walls of the container are perfectly insulating, what is the final temperature T_f?

(d) Now suppose that the container is filled with a gas of electrons and positrons, again with negligible interactions and no chemical potential. Denoting the energy of a single particle by $E_e(|\vec{q}|)$, where $\vec{q}$ is its momentum, redo part (a) for this case. That is, write an expression for the free energy $F_e(L,T)$ of the electron-positron gas in the form

$$F_e(L,T) = \int d^3q\, f_e(|\vec{q}|) .$$

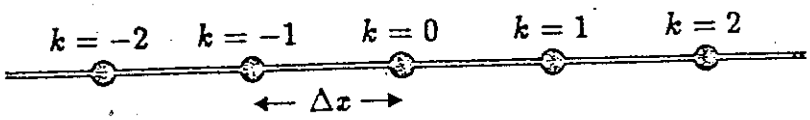
(e) If $kT \gg m_e c^2$, where $m_e c^2$ is the rest energy of an electron, then the rest energy will be negligible and the electron-positron gas will behave as if the particles were massless. In that case the integration can be carried out, with the result

$$F_e(L,T) = -\frac{7\pi^2}{180}\frac{(kT)^4}{(\hbar c)^3} L^3 .$$

Now suppose initially the box is heated to a temperature $kT \ll m_e c^2$, so that only photons contribute significantly to the contents. As in part (c), the box is slowly compressed so that the sides have length αL, with no heat lost to the walls. This time, however, assume that the final temperature satisfies $kT_f \gg m_e c^2$, so that electron-positron pairs contribute copiously (along with photons) to the final mix. Find the final temperature T_f for this case.

Statistical Mechanics Problem 2
Diffusion on a Lattice

A particle moves on a one-dimensional lattice, with lattice sites labeled by an integer k. After each successive time interval Δt, the particle jumps one unit to the left or right (i.e., $\Delta k = \pm 1$), with each direction having equal probability.



Let $q(k,t)$ denote the probability that at time t the particle is found at lattice site k. Let Δx denote the physical distance between the lattice sites, so the site k is located at

$$x = k\,\Delta x \ .$$

Suppose that the probability $q(k,t)$ for the particle to be at the site k varies slowly from one site to the next, so the probability can be approximated by a continuous probability density (probability per unit length) $P(x,t)$, where

$$q(k,t) = P(x,t)\,\Delta x\big|_{x=k\,\Delta x} \ .$$

Assume that $P(x,t)$ is a smooth function for all x, with a Taylor expansion about any point x_0 which converges rapidly for $|x - x_0| \lesssim \Delta x$.

(a) By considering what happens during a single time step Δt, show that the evolution can be described by a diffusion equation of the form

$$\frac{\partial P(x,t)}{\partial t} = D\,\frac{\partial^2 P(x,t)}{\partial x^2} \ ,$$

where D is called the diffusion coefficient. Calculate D in terms of Δx and Δt.

(b) Now suppose that at each time step Δt the particle can jump any number of steps Δk, restricted by $|\Delta k| \le 5$. The probability for any particular value of Δk is given by a function $p(\Delta k)$, with

$$\sum_{\Delta k=-5}^{5} p(\Delta k) = 1 \ .$$

Assuming that $P(x,t)$ varies slowly over 5 lattice sites, show that it obeys a generalized diffusion equation

$$\frac{\partial P(x,t)}{\partial t} = \alpha\,\frac{\partial P(x,t)}{\partial x} + D\,\frac{\partial^2 P(x,t)}{\partial x^2} \ ,$$

where α is a constant. Express the constants α and D in terms of Δx, Δt, and the function $p(\Delta k)$.

— PROBLEM CONTINUES ON NEXT PAGE —

(c) Now suppose that the one-dimensional lattice with sites labeled by the integer k is replaced by a three-dimensional lattice, with the sites labeled by the triplet of integers $(k_x, k_y, k_z) \equiv \vec{k}$. At each time step Δt the particle has an equal probability of making any of the six jumps $\Delta k_x = 1$, $\Delta k_x = -1$, $\Delta k_y = 1$, $\Delta k_y = -1$, $\Delta k_z = 1$, or $\Delta k_z = -1$. The lattice spacing in all three directions is Δx, so the spatial coordinates are given by

$$x = k_x \, \Delta x$$

$$y = k_y \, \Delta x$$

$$z = k_z \, \Delta x \ ,$$

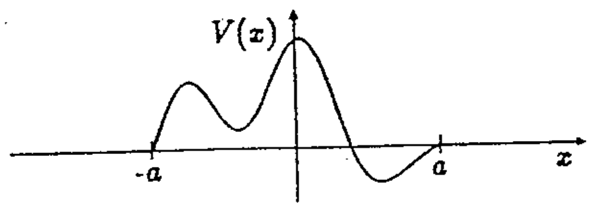
and the probability density (probability per unit volume) is related by

$$q(\vec{k}, t) = P(\vec{x}, t) \, \Delta x^3 \big|_{\vec{x} = \vec{k} \, \Delta x} \ .$$

Write the appropriate diffusion equation for the evolution of $P(\vec{x}, t)$ in this case, and evaluate the diffusion coefficient in terms of Δx and Δt.

Quantum Mechanics Problem 1
Particle Motion in One Dimension

Consider a particle of mass m which moves in one dimension under the influence of a potential $V(x)$. $V(x)$ vanishes for $x \le -a$ and for $x \ge a$, where a is a constant, but $V(x)$ might have any form in the region $-a < x < a$.



The Schrödinger wave function for the particle will be denoted by $\Psi(x,t)$.

(a) Use the time-dependent Schrödinger equation to derive an equation for the conservation of probability of the form

$$\frac{\partial \rho}{\partial t} = -\frac{\partial j}{\partial x} ,$$

where $\rho(x,t) \equiv |\Psi(x,t)|^2$. Give an explicit expression for the probability current $j(x,t)$ in terms of $\Psi(x,t)$, m, and $\hbar$. Do not set $\hbar$ equal to one.

(b) Now consider a scattering situation, for which $\Psi(x,t) = \psi_1(x)e^{-i\omega t}$, and $\psi_1(x)$ has the property that

$$\psi_1(x) = e^{ikx} + Re^{-ikx} \qquad \text{for } x < -a$$

$$\psi_1(x) = Te^{ikx} \qquad \text{for } x > a .$$

Here k is a constant, and the constants R and T are called the reflection and transmission amplitudes, respectively. Use the conservation of probability to show that

$$|R|^2 + |T|^2 = 1 .$$

(c) Now consider a wave function that describes scattering from the right, instead of from the left. Thus $\Psi(x,t) = \psi_2(x)e^{-i\omega t}$, where

$$\psi_2(x) = e^{-ikx} + R'e^{ikx} \qquad \text{for } x > a$$

$$\psi_2(x) = T'e^{-ikx} \qquad \text{for } x < -a .$$

Show that

$$T' = T$$

and

$$R' = -\frac{R^*T}{T^*} .$$

(Hint: try to construct $\psi_2(x)$ as a linear combination of $\psi_1(x)$ and $\psi_1^*(x)$.)

(d) Now suppose that $V(x)$ is symmetric, so $V(-x) = V(x)$. For this special case, show that the product R^*T is a purely imaginary number.

Quantum Mechanics Problem 2
Quantum "Teleportation" of Spin-$\frac{1}{2}$ Particles

In this problem we will be considering the spin states of spin-$\frac{1}{2}$ particles. The spin-up ($s_z = +\frac{1}{2}\hbar$) and spin-down ($s_z = -\frac{1}{2}\hbar$) states will be denoted by $|+\rangle$ and $|-\rangle$, respectively.

(a) Consider first a system with two spin-$\frac{1}{2}$ particles, labeled particle 1 and particle 2. The system can be described in terms of the basis states $|++\rangle_{12}$, $|--\rangle_{12}$, $|+-\rangle_{12}$, and $|-+\rangle_{12}$, where the subscripts indicate which particles are being described by the "+" and "−" symbols inside the ket. Consider the operator Q described by the matrix

$$Q = \begin{array}{c} \\ {}_{12}\langle ++| \\ {}_{12}\langle --| \\ {}_{12}\langle +-| \\ {}_{12}\langle -+| \end{array} \begin{array}{c} \begin{array}{cccc} |++\rangle_{12} & |--\rangle_{12} & |+-\rangle_{12} & |-+\rangle_{12} \end{array} \\ \left(\begin{array}{cccc} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right) \end{array}$$

Find the eigenvalues and eigenvectors of this operator.

(b) Suppose that Alice prepares two spin-$\frac{1}{2}$ particles in a spin-0 state. The two particles are labeled 2 and 3, as the label 1 is being saved for later use. Thus the two-particle state vector can be written

$$|\Psi\rangle_{23} = \frac{1}{\sqrt{2}}\left(|+-\rangle_{23} - |-+\rangle_{23}\right) .$$

Without disturbing the spins of either particle, particle 3 is given to Bob, who carries it to some other location. Particles 2 and 3 are called an "entangled pair."

Alice is then given a third spin-$\frac{1}{2}$ particle, particle 1, in an arbitrary quantum state denoted by

$$|\Phi\rangle_1 = a|+\rangle_1 + b|-\rangle_1 ,$$

where $|a|^2 + |b|^2 = 1$. Her goal is to "teleport" this quantum state to Bob, making use of the entangled pair. Note that she has not been told the coefficients a and b, and that she cannot determine them by any experiment performed on the single spin-$\frac{1}{2}$ particle that she was given.

Since particle 1 is uncorrelated with particles 2 or 3, the full quantum state of the three particles can be written as a product state:

$$\begin{aligned} |\Omega\rangle_{123} &= |\Phi\rangle_1 |\Psi\rangle_{23} \\ &= \frac{a}{\sqrt{2}}\left(|++-\rangle_{123} - |+-+\rangle_{123}\right) + \frac{b}{\sqrt{2}}\left(|-+-\rangle_{123} - |--+\rangle_{123}\right) . \end{aligned}$$

With the particles in the state $|\Omega\rangle_{123}$ given above, suppose Alice measures the spin of particle 1. What is the probability p_1 that it is up (+)? Suppose instead she measures the spin of particle 2. What is the probability p_2 that it is up?

— PROBLEM CONTINUES ON NEXT PAGE —

(c) Suppose now that Alice has not made either of the two measurements discussed in part (b), so the three particles are still in the state described above as $|\Omega\rangle_{123}$. Using particles 1 and 2, Alice measures the operator Q defined in part (a). What is the probability p_{-1} that she obtains the result $Q = -1$? (Hint: try to rewrite $|\Omega\rangle_{123}$ in a basis in which particles 1 and 2 are described as eigenstates of Q, while particle 3 is described as an eigenstate of s_z.)

(d) Suppose that Alice does obtain the result $Q = -1$, and she communicates this result to Bob. What is the state $|\Omega'\rangle_{123}$ of the three-particle system after the measurement? Show that Bob finds particle 3 in a spin eigenstate, which up to an irrelevant phase is identical to the state $|\Phi\rangle$ that Alice was trying to transmit. (Note for the curious: if Alice had obtained a different result, particle 3 would not have been in the state $|\Phi\rangle$. In all cases, however, particle 3 would be in a spin eigenstate which is a rotation of $|\Phi\rangle$. By learning the result of Alice's experiment, Bob would know what rotation to perform on particle 3 so that it could be restored to the state $|\Phi\rangle$.)

[This problem is based on C.H. Bennett et al., "Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels," *Phys. Rev. Lett.* 70, 1895 (1993).]