MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS
EDUCATION OFFICE
ROOM 4-352
PHONE (617)253-4842
FAX (617) 258-8319

DOCTORAL GENERAL EXAMINATION

PART II

February 4, 2000

FIVE HOURS

1. This examination is divided into four sections, Mechanics, Electricity & Magnetism, Statistical Mechanics, and Quantum Mechanics, with two problems in each. It is advisable to carefully read both problems in each section before making your choice. Submit ONLY one problem per section. IF YOU SUBMIT MORE THAN ONE PROBLEM FROM A SECTION, BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.

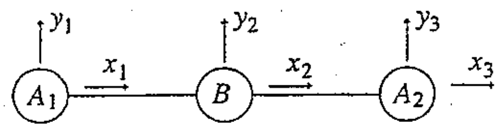
2. Use a separate fold of paper for each problem, and write your name on each fold. Include the problem number with each solution.

3. Calculators may be used.

4. **No Books or Reference Materials May Be Used.**

Mechanics Problem 1

Consider small vibrations of a molecule made up of two identical atoms A having mass m_A, denoted A_1 and A_2 in the drawing, and a central atom B having mass m_B. Assume the potential energy depends only on the distances $\overline{A_1B}$, $\overline{BA_2}$, and the angle $\angle A_1BA_2$.



a) Determine the number of longitudinal and transverse vibrational modes for fully three-dimensional motion. Explain why all the frequencies may be determined by considering vibrations in the x-y plane.

b) Denote the displacements from equilibrium of each atom x_i, y_i as shown in the drawing. Eliminate three variables by working in a coordinate system such that the total momenta $P_x = P_y = 0$ and the total angular momentum $J_z = 0$, assuming that the displacements are small.

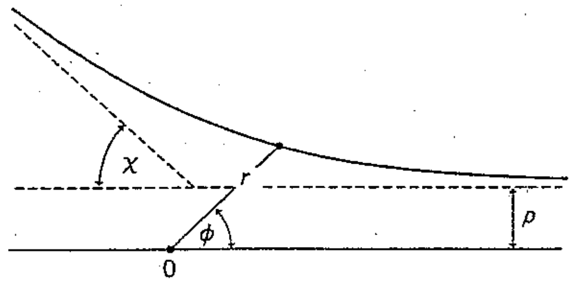
c) Show that the potential energy for small oscillations has the form

$$U = \frac{\kappa}{2}(x_3 - x_2)^2 + \frac{\kappa}{2}(x_1 - x_2)^2 + \frac{\alpha}{2}(y_1 + y_3 - 2y_2)^2.$$

d) Calculate the normal modes and corresponding vibrational frequencies. Sketch how the atoms move for each mode.

Mechanics Problem 2

Consider the scattering of a beam of particles by a repulsive central force $F = kr^{-3}$.



a) Define the differential cross section for classical scattering, $\frac{d\sigma}{d\Omega}$.

b) Show that $\frac{d\sigma}{d\Omega} = \frac{\rho}{\sin\chi}\left(\frac{d\rho}{d\chi}\right)$ where χ is the deflection angle relative to the forward direction and ρ is the impact parameter as shown in the drawing.

c) Using polar coordinates (ϕ, r) for the particle moving in the central potential $U(r)$ with $U(r \to \infty) = 0$, show that

$$d\phi = \frac{\frac{\rho}{r^2}\,dr}{\sqrt{1 - \frac{U(r)}{E} - \frac{\rho^2}{r^2}}}\,.$$

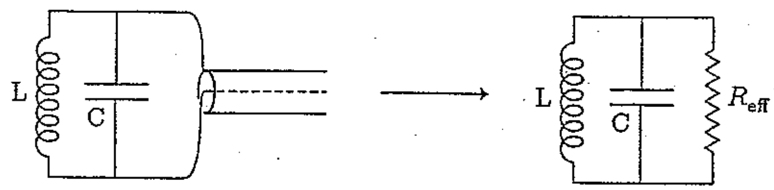
d) Calculate $\frac{d\sigma}{d\Omega}$ for the repulsive force $F = kr^{-3}$. The integral $\int \frac{dr}{r\sqrt{r^2-1}} = \cos^{-1}\left(\frac{1}{r}\right)$ may be useful.

Electromagnetism Problem 1

PART 1. Consider the TEM mode in a coaxial cable, consisting of two perfectly conducting cylinders of radii a and b ($a < b$). Clearly state your choice of units in answering this problem.

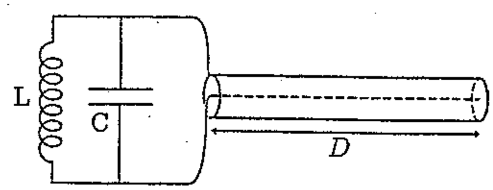
a) For the TEM mode with a wavenumber k in an infinite cable, find the E and B fields in the space between the cylinders, and the charge and current densities on the surface of each cylinder. Make a drawing.

b) Demonstrate that at low frequencies ($\omega \ll c/b$) the input to a semi-infinite cable is represented by an effective resistor (also called "the cable input impedance"), so that $V_{\rm ab} = R_{\rm eff}\, I_{\rm ab}$. Find the value of $R_{\rm eff}$.



c) The semi-infinite cable described in part (b) is connected across an L-C circuit, as shown in the drawing. One has to design the damped LC oscillator so that it has a quality factor $Q > 100$. Derive the corresponding condition for L and C in terms of the cable geometric parameters a and b.

d) Now consider the LC circuit of question (c) coupled to a finite segment of the coaxial cable of length $D \gg b, a$K. , as shown in the drawing below. The far end of the cable is open. Derive a characteristic equation for the resonance frequencies of the system.



PART 2. In an infinite coaxial cable, the outer radius b slowly varies along the cable, from $b = b_1$ to $b = b_2$, where $b_1 > b_2$, over a scale distance d. For instance:

$$b(x) = \frac{b_1 e^{-x/2d} + b_2 e^{+x/2d}}{e^{-x/2d} + e^{+x/2d}} .$$

An E&M wave propagating along the cable from $x = -\infty$ will be partially reflected from the inhomogeneity region, and partially transmitted. Find the reflection coefficient $R = |E_{\rm refl}/E_{\rm inc}|^2$ for the TEM mode of wavelength $\lambda \gg d$.

Electromagnetism Problem 2

A magnetic field is given by

$$B_x = 0\ , \quad B_y = B_o \frac{x}{\sqrt{x^2+\lambda^2}}\ , \quad B_z = 0$$

(Note that such a field cannot exist in a vacuum. You may assume that the field is generated by a current of electrons, but you should neglect the electrons in this problem.) A nonrelativistic proton moves in this field under the action of the Lorentz force.

a) What are the constants of the motion for the proton in this configuration?

b) What kind of orbit can the proton have for $x^2 \gg \lambda^2$, assuming that its energy $\epsilon \ll m_p c^2$, and that $v_\perp/(eB_0/m_p c) \ll \lambda$? Here $v_\perp = \sqrt{v_z^2 + v_x^2}$ is the component of the velocity perpendicular to the magnetic field, and $v_\parallel = v_y$ is the parallel component. Both $v_\perp$ and $v_\parallel \neq 0$.

c) For the same conditions as in part (b), what kind of orbit can a proton have in the region near $x = 0$ where $x^2 < \lambda^2$ and $B_y \simeq B_o x/\lambda$?

d) Find the order of magnitude of the size of the region near $x = 0$ where the topology of the orbits, with $0 < v_\perp(eB_o\lambda/m_p c) \ll 1$, is drastically different from that which is characteristic of the "far" region $x^2 \gg \lambda^2$.

Statistical Mechanics Problem 1

Consider N spins-1/2 at temperature T in an external magnetic field with Hamiltonian

$$H_0 = -\mu B \sum_{i=1}^{N} s_i^z \, .$$

a) Find the partition function of the system $Z_0(B, \beta)$, where $\beta = 1/k_B T$.

b) Find the free energy F and the entropy S.

Now, suppose that the spins are interacting, so that the total Hamiltonian is $H = H_0 + H_{\text{int}}$. The interaction is the same between all spins and is given by

$$H_{\text{int}} = -J \sum_{i<j} s_i^z s_j^z \, .$$

The coupling constant J is positive, corresponding to a ferromagnet.

c) Show that the partition function of the system can be written

$$Z = A \int_{-\infty}^{\infty} Z_0(B + \lambda/\mu, \beta) e^{-\beta\lambda^2/2J} \, d\lambda \qquad (1)$$

and find the constant A.

d) Consider a large number of spins $N \gg 1$. In this "macroscopic" limit the partition function can be analyzed using the so-called "saddle-point approximation," which amounts to writing the integrand in Eq. (1) as $e^{-\beta F(\lambda)}$, looking for the minima of the function $F(\lambda)$, and evaluating the expression near the minima as a Gaussian integral.

Consider the function $F(\lambda)$ for $B = 0$. Show that at high temperature $F(\lambda)$ has only one minimum at $\lambda = 0$. Plot $F(\lambda)$ for different temperatures. Find the critical temperature T_c below which new minima appear. Interpret the temperature T_c in terms of spin ordering.

e) Find the linear magnetic susceptibility at $T > T_c$.

Statistical Mechanics Problem 2

Consider a one-dimensional quantum mechanical oscillator with mass m and spring constant k. The system is in thermodynamic equilibrium characterized by temperature T.

a) Write the system energy levels E_n and calculate the partition function Z.

b) Find the mean energy $\langle E \rangle$ of the system and specific heat C as a function of temperature.

Now, consider a string of length L, having density ρ per unit length. The string is stretched with tension F_0, and its ends are fixed to hard walls. The state of the string is characterized by a transverse displacement field $\mathbf{u}(x,t)$ where x is the coordinate along the string $(0 \leq x \leq L)$.

c) Write the classical equation of motion of the string, and look for solutions with the boundary conditions $u(x=0) = u(x=L) = 0$. Find the normal modes $U_m(x)$ of the system $(m = 1, 2, 3 \ldots)$. (Don't forget that there are two transverse polarization directions for the string displacement.)

d) Show that the potential energy of the string can be written as $\int_0^L \frac{1}{2} F_0 (\partial u/\partial x)^2 \, dx$. Write the classical Hamiltonian in terms of the normal mode amplitudes defined by the generalized Fourier series $u(x) = \sum_m q_m U_m(x)$.

e) The quantum string problem can be solved by associating a quantum mechanical oscillator with each normal mode of string oscillation. Write the Hamiltonian operator in terms of q_m. In the limit of a long string, $L \gg \hbar c / k_B T$, find the mean energy of the system and the specific heat as functions of temperature.

Quantum Mechanics – Problem 1

Consider a Dirac particle in a uniform magnetic field of magnitude B in the z direction. The Hamiltonian may be written

$$\widehat{H} = c\boldsymbol{\alpha} \cdot \boldsymbol{\pi} + \beta mc^2$$

where

$$\boldsymbol{\pi} = \mathbf{p} - \frac{q}{c}\mathbf{A}$$
$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}.$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and σ_i denotes the Pauli matrices.

a) Work in the gauge $A_z = 0$ and calculate the commutators $[\pi_i, \pi_j]$. Note that one of them should involve B. The anticommutation relations $\{\alpha_i, \alpha_j\}$ and $\{\alpha_i, \beta\}$ may be useful.

b) Calculate H^2 and show that it can be written as the sum of three commuting terms

$$h_1 = c^2(\pi_x^2 + \pi_y^2)$$
$$h_2 = c^2(\pi_z^2)$$
$$h_3 = \alpha_i \alpha_j [\pi_i, \pi_j]$$

plus a constant.

c) Find the eigenvalues of the three commuting operators and thus find the eigenvalues of H^2.

d) Write the eigenvalues of H and calculate the nonrelativistic limit to order $1/m$.

Quantum Mechanics – Problem 2

Three noninteracting identical particles of mass m are in a three-dimensional box of dimensions $L \times L \times L/3$:

$$V = \begin{cases} 0\,, & 0 < x < L, 0 < y < L, 0 < z < L/3 \\ \infty\,, & \text{otherwise.} \end{cases}$$

a) Calculate the three lowest single-particle energy eigenvalues and all the corresponding eigenfunctions. Express all energies in units of $\epsilon_0 = \hbar^2\pi^2/2mL^2$.

b) First, assume the particles have spin 0. For the combined system of three particles, by considering all states of the appropriate symmetry find the two lowest eigenvalues of the total energy and their degeneracies (i.e., the total number of distinct states of the proper symmetry that have each energy).

c) Repeat part (b) for spin-1/2 particles.

d) Repeat part (b) for spin-1 particles.

In your exam book, summarize the results for (b), (c), and (d) in a table of the following form:

	Ground State		First Excited State	
Spin	Energy	Degeneracy	Energy	Degeneracy
0				
1/2				
1				