General Exam Spring 2001 Solutions

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Mechanics Problem 1

(1a) Let $(x_{\rm D}, y_{\rm D})$ be the coordinates of the center of mass of m_1, and let $(x_{\rm C}, y_{\rm C})$ be the coordinates of m_2. The angles φ_1 and φ_2 are defined by the figure. The kinetic energy of the system is given by

$$T = \frac{m_1}{2}\left(\dot{x}_{\rm D}^2 + \dot{y}_{\rm D}^2\right) + \frac{\mathcal{I}}{2}\dot{\varphi}_1^2 + \frac{m_2}{2}\left(\dot{x}_{\rm C}^2 + \dot{y}_{\rm C}^2\right) \tag{1}$$

and its potential energy is given by

$$V = -g\left(m_1 y_{\rm D} + m_2 y_{\rm C}\right). \tag{2}$$

The Lagrangian $\mathcal{L}$ is given by $\mathcal{L} = T - V$; however, we have been asked to express $\mathcal{L}$ in terms of φ_1 and φ_2 and their derivatives. These are related to the Cartesian coordinates by

$$\begin{cases} x_{\rm D} = a\sin\varphi_1 \\ y_{\rm D} = a\cos\varphi_1 \\ x_{\rm C} = b\sin\varphi_1 + c\sin\varphi_2 \\ y_{\rm C} = b\cos\varphi_1 + c\cos\varphi_2 \end{cases} \tag{3}$$

which give derivatives

$$\begin{cases} \dot{x}_{\rm D} = a\dot{\varphi}_1\cos\varphi_1 \\ \dot{y}_{\rm D} = -a\dot{\varphi}_1\sin\varphi_1 \\ \dot{x}_{\rm C} = b\dot{\varphi}_1\cos\varphi_1 - c\dot{\varphi}_2\cos\varphi_2 \\ \dot{y}_{\rm C} = -b\dot{\varphi}_1\cos\varphi_1 - c\dot{\varphi}_2\cos\varphi_2 \end{cases} \tag{4}$$

Plugging (3) and (4) into (1) and (2), we find

$$T = \frac{\dot{\varphi}_1^2}{2}\left(m_1 a^2 + \mathcal{I} + m_2 b^2\right) + \frac{1}{2}m_2 c^2\dot{\varphi}_2^2 + m_2 bc\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1 - \varphi_2) \tag{5}$$

(recall $\cos(\varphi_1 - \varphi_2) = \cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2$) and

$$V = -g\cos\varphi_1\left(m_1 a + m_2 b\right) - m_2 gc\cos\varphi_2 \tag{6}$$

therefore

$$\boxed{\mathcal{L} = \frac{\dot{\varphi}_1^2}{2}\left(m_1 a^2 + \mathcal{I} + m_2 b^2\right) + \frac{1}{2}m_2 c^2\dot{\varphi}_2^2 + m_2 bc\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1 - \varphi_2) + g\cos\varphi_1\left(m_1 a + m_2 b\right) + m_2 gc\cos\varphi_2} \tag{7}$$

(1b) The Euler-Lagrange equations are

$$\begin{cases} \dfrac{\partial\mathcal{L}}{\partial\varphi_1} = \dfrac{d}{dt}\left(\dfrac{\partial\mathcal{L}}{\partial\dot{\varphi}_1}\right) \\ \dfrac{\partial\mathcal{L}}{\partial\varphi_2} = \dfrac{d}{dt}\left(\dfrac{\partial\mathcal{L}}{\partial\dot{\varphi}_2}\right) \end{cases} \tag{8}$$

These give the equations of motion

$$\boxed{\begin{cases} \left(m_1 a^2\mathcal{I} + m_2 b^2\right)\ddot{\varphi}_1 + m_2 bc\ddot{\varphi}_2\cos(\varphi_1 - \varphi_2) + m_2 bc\dot{\varphi}_2^2\sin(\varphi_1 - \varphi_2) = -(m_1 a + m_2 b)g\sin\varphi_1 \\ m_2 c^2\ddot{\varphi}_2 + m_2 bc\ddot{\varphi}_1\cos(\varphi_1 - \varphi_2) - m_2 bc\dot{\varphi}_1^2\sin(\varphi_1 - \varphi_2) = -m_2 cg\sin\varphi_2 \end{cases}} \tag{9}$$

(1c) For both pieces to move as one rigid body, we require $\varphi_1 = \varphi_2 \equiv \varphi$. Plugging this constraint into (9) and simplifying, we arrive at

$$\begin{cases} \left(m_1 a^2 + \mathcal{I} + m_2 b^2 + m_2 bc\right) \ddot{\varphi} = -\left(m_1 a + m_2 b\right) g \sin\varphi \\ \left(m_2 c^2 + m_2 bc\right) \ddot{\varphi} = -m_2 c g \sin\varphi \end{cases} \tag{10}$$

Dividing one equation of (10) by the other and cancelling some common factors, we find

$$\boxed{\frac{b+c}{m_1 a^2 + m_2 b^2 + m_2 bc + \mathcal{I}} = \frac{1}{m_1 a + m_2 b}} \tag{11}$$

Mechanics Problem 2

(a) Let x_1, x_2, and x_3 denote the displacements of the masses from their equilibrium positions. (Note: if you want to calculate the equations of motion for each mass using a Newtonian or Hamiltonian approach, please feel free to skip this Lagrangian derivation.) The Lagrangian for this system is

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right) - \frac{1}{2}k\left[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2\right] \tag{12}$$

and applying the Euler-Lagrange equations gives the equations of motion

$$\begin{cases} m\ddot{x}_1 = -2kx_1 + kx_2 + kx_3 \\ m\ddot{x}_2 = kx_1 - 2kx_2 + kx_3 \\ m\ddot{x}_3 = kx_1 + kx_2 - 2kx_3 \end{cases} \tag{13}$$

Packaging the previous equation as a 3×3 matrix, we find

$$m\frac{d^2}{dt^2}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2k & k & k \\ k & -2k & k \\ k & k & -2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{14}$$

For small oscillations, we expect the motion to be simple-harmonic with characteristic frequency ω, so we take as our *ansatz* for $\mathbf{u} \equiv (x_1, x_2, x_3)^\mathsf{T}$ a solution of the form

$$\mathbf{x} = \mathbf{A}e^{i\omega t}, \tag{15}$$

where $\mathbf{A}$ is a (possibly complex-valued) constant vector describing the initial conditions for each mass. Plugging this in, we find

$$\begin{bmatrix} 2k - m\omega^2 & -k & -k \\ -k & 2k - m\omega^2 & -k \\ -k & -k & 2k \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{16}$$

We recognize this as a standard eigenvalue problem. Finding the eigenvalues (normal-mode frequencies) of the 3×3 matrix using $\det(\mathsf{M} - m\omega^2\mathbf{1})$, we have the characteristic equation

$$\left(\frac{m\omega^2}{k}\right)\left(\frac{m\omega^2}{k} - 3\right) = 0 \tag{17}$$

which gives

$$\boxed{\omega_1 = 0, \qquad \omega_2 = \omega_3 = \sqrt{\frac{3k}{m}}} \tag{18}$$

Case 1: $\omega = 0$

The eigenvector equation corresponding to this eigenvalue is

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

from which we find $A_1 = A_2 = A_3$, giving the (normalized) eigenvector

$$\mathbf{A} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{20}$$

We need to remember, however, that $\omega = 0$ implies $e^{i\omega t} = 1$, so we need to figure out something else for the time dependence. We note that $\omega = 0$ corresponds to all of the masses maintaining a constant separation from each other, which implies that the ring and beads are rotating as a rigid body with a constant speed. Thus[1] we have

$$\mathbf{A} = \frac{1}{\sqrt{3}}(at + b)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{21}$$

[1] You could also solve the ODE $\ddot{x} = 0$ by inspection.

Case 2: $\omega = \sqrt{3k/m}$
In this case, the eigenvector equation is

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{22}$$

which does not uniquely determine both orthogonal eigenvectors we need. If we set $A_3 = 0$, then we find one of the eigenvectors (including the time dependence)

$$\mathbf{A} = \frac{1}{\sqrt{2}} e^{i(\omega t + \phi)} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \tag{23}$$

where $\phi \in \mathbb{R}$ is some constant. (This is an overall undetermined complex phase factor, which will turn out to be exactly the phase we need to specify the initial conditions for our sines and cosines.) To find the other eigenvector, we can either use the technique of *generalized eigenvectors*

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \frac{1}{\sqrt{2}} e^{i(\omega t + \phi)} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \tag{24}$$

or use the Gram-Schmidt procedure; either way, we get

$$\mathbf{A} = \frac{1}{\sqrt{6}} e^{i(\omega t + \phi)} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \tag{25}$$

Thus, we can write the positions of the masses as a linear combination of the normal modes (here, assuming that the $C_i \in \mathbb{R}$):

$$\boxed{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = (C_1 t + C_2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (C_3 \sin(\omega t) + C_4 \cos(\omega t)) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + (C_5 \sin(\omega t) + C_6 \cos(\omega t)) \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}} \tag{26}$$

(Here, I exchanged $Ce^{i(\omega t + \phi)}$ for $A \sin(\omega t) + B \cos(\omega t)$. The two approaches are equivalent; I just prefer not working with complex phases as initial conditions.)

(c) We have the initial conditions $x_1(0) = \delta$, $x_2(0) = x_3(0) = 0$, and $\dot{x}_1(0) = \dot{x}_2(0) = \dot{x}_3(0) = 0$. This gives us

$$\begin{cases} \text{(i):} & \delta = C_2 + C_4 + C_6 \\ \text{(ii):} & 0 = C_2 - C_4 + C_6 \\ \text{(iii):} & 0 = C_2 - 2C_6 \end{cases}$$

$$\begin{cases} \text{(iv):} & 0 = C_1 + \omega(C_3 + C_5) \\ \text{(v):} & 0 = C_1 + \omega(-C_3 + C_5) \\ \text{(vi):} & 0 = C_1 - 2\omega C_5 \end{cases}$$

From these, we quickly see that $C_3 = C_5 = 0$, so our solution has cosines instead of sines. We also find $C_1 = 0$, as well as $C_4 = \delta/2$, $C_2 = \delta/3$, and $C_6 = \delta/6$. Thus our solution is

$$\boxed{\begin{cases} x_1(t) = \dfrac{\delta}{3} + \dfrac{2\delta}{3} \cos(\omega t) \\ x_2(t) = \dfrac{\delta}{3} - \dfrac{\delta}{3} \cos(\omega t) \\ x_3(t) = \dfrac{\delta}{3} - \dfrac{\delta}{3} \cos(\omega t) \end{cases}} \tag{27}$$

Electromagnetism Problem 1

(a) The $\vec{D}$- and $\vec{B}$-fields in the medium are

$$\begin{cases} \vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \gamma \vec{\nabla} \times \vec{E} \\ \vec{B} = \mu_0 \vec{H} + \cancel{\mu_0 \vec{M}} \end{cases} \tag{28}$$

($\vec{M} = 0$ since the medium is non-magnetic). From the curl equations,

$$\begin{cases} \vec{\nabla} \times \vec{B} = \mu_0 \dfrac{\partial \vec{D}}{\partial t} = \mu_0 \epsilon_0 \dfrac{\partial \vec{E}}{\partial t} + \mu_0 \gamma \vec{\nabla} \times \left(\dfrac{\partial \vec{E}}{\partial t} \right) \\ \vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \end{cases}. \tag{29}$$

Take the curl of both sides of Faraday's law:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E} \right) = -\frac{\partial}{\partial t} \left(\vec{\nabla} \times \vec{B} \right) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} - \gamma \mu_0 \vec{\nabla} \times \left(\frac{\partial^2 \vec{E}}{\partial t^2} \right) \tag{30}$$

From vector analysis, we have the general result

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{V} \right) = -\vec{\nabla}^2 \vec{V} + \vec{\nabla} \left(\vec{\nabla} \cdot \vec{V} \right). \tag{31}$$

In this problem, $\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \vec{B} = 0$, so (21) simplifies to (using $c = 1/\sqrt{\mu_0 \epsilon_0}$)

$$-\vec{\nabla}^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} - \gamma \mu_0 \vec{\nabla} \times \left(\frac{\partial^2 \vec{E}}{\partial t^2} \right) \tag{32}$$

For a plane wave propagating in the z-direction, we have

$$\vec{E} = \vec{E}_0 e^{i(kz - \omega t)}. \tag{33}$$

Plugging this into (23), we find

$$k^2 \vec{E} = \frac{\omega^2}{c^2} \vec{E} + i \gamma \mu_0 \omega^2 k \left(\hat{\mathbf{z}} \times \vec{E} \right) \tag{34}$$

As the wave is propagating in the z-direction, only the E_x and E_y components need be considered:

$$\begin{cases} \left(k^2 - \dfrac{\omega^2}{c^2} \right) E_x + i \gamma \mu_0 \omega^2 k E_y = 0 \\ \left(k^2 - \dfrac{\omega^2}{c^2} \right) E_y - i \gamma \mu_0 \omega^2 k E_x = 0 \end{cases} \tag{35}$$

These are nonvanishing only if

$$\left(k^2 - \frac{\omega^2}{c^2} \right)^2 - \gamma^2 \mu_0^2 \omega^4 k^2 = 0 \tag{36}$$

which simplifies to

$$k^2 \mp \gamma \mu_0 \omega^2 k - \frac{\omega^2}{c^2} = 0 \tag{37}$$

$$k = \frac{1}{2} \left[\pm \mu_0 \gamma \omega^2 \pm \sqrt{\gamma^2 \mu_0^2 \omega^4 + \frac{4\omega^2}{c^2}} \right] \tag{38}$$

Choose the $k > 0$ solution (the quantity inside the square root is strictly positive, and $\sqrt{\dots} > \mu_0 \gamma \omega^2$):

$$k_\pm = \frac{1}{2} \left[\pm \gamma \mu_0 \omega^2 + \sqrt{\gamma^2 \mu_0^2 \omega^4 + \frac{4\omega^2}{c^2}} \right] \tag{39}$$

The index of refraction $n = ck/\omega$, so

$$n_\pm = \frac{ck_\pm}{\omega} = \pm\frac{\gamma c \mu_0 \omega}{2} + \sqrt{1 + \frac{\gamma^2 c^2 \mu_0^2 \omega^2}{2}} \tag{40}$$

$$\boxed{n_\pm \approx 1 \pm \frac{\gamma c \mu_0 \omega}{2}} \tag{41}$$

(b) We can package (35) as a matrix equation:

$$\begin{bmatrix} k^2 - \frac{\omega^2}{c^2} & i\gamma\mu_0\omega^2 k \\ -i\gamma\mu_0\omega^2 k & k^2 - \frac{\omega^2}{c^2} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{42}$$

We want to find electric field configurations which have the dispersion relations (and hence indices of refraction) we found above. In other words, we want to find the eigenstates of this matrix. The eigenvalues are

$$(k^2 - \omega^2) \pm \gamma\mu_0 k\omega^2 \tag{43}$$

which correspond to the eigenvectors $(\pm i, 1)^\mathsf{T}$. These are circularly-polarized waves:

$$\boxed{\begin{cases} E_+ = \dfrac{1}{\sqrt{2}}(iE_x + E_y) \\ E_- = \dfrac{1}{\sqrt{2}}(-iE_x + E_y) \end{cases}} \tag{44}$$

(Note: there are many different conventions for polarized EM waves. The crucial part is the relative factor of $(-)i$ between E_x and E_y.)

Electromagnetism Problem 2

(a) As $r \to \infty$, the electric field becomes a constant $E_0\hat{\mathbf{z}}$. In spherical coordinates, this corresponds to

$$\vec{E}(r \to \infty) = E_0 \left(\hat{\mathbf{r}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta\right) \tag{45}$$

The scalar potential V that satisfies the given boundary conditions is

$$V(r,\theta) = \frac{A}{r} + \left(Br + \frac{C}{r^2}\right)\cos\theta \tag{46}$$

for undetermined constants A, B, and C. We can use this to calculate the $\vec{E}$-field:

$$\vec{E} = -\vec{\nabla} V = \frac{A}{r^2}\hat{\mathbf{r}} - \left(B - \frac{2C}{r^3}\right)\cos\theta\hat{\mathbf{r}} + \left(B + \frac{C}{r^3}\right)\sin\theta\hat{\boldsymbol{\theta}} \tag{47}$$

To satisfy the boundary condition as $r \to \infty$, we require $B = -E_0$. The surface of the sphere is an equipotential $V(R) = \text{const}$ (independent of θ) so

$$BR + \frac{C}{R^2} = 0 \tag{48}$$

$$\Rightarrow C = E_0 R^3 \tag{49}$$

The A coefficient is simply related to the total charge Q_0 on the sphere:

$$A = \frac{Q_0}{4\pi\epsilon_0} \tag{50}$$

To prove this, you can use Gauss' law in integral form:

$$\int \vec{E} \cdot d\vec{A} = Q_0/\epsilon_0 \tag{51}$$

$$\int_{\phi=0}^{2\pi} \int_{\cos\theta=-1}^{1} (\vec{E} \cdot \hat{\mathbf{r}}) r^2 \sin\theta \, d(\cos\theta) \, d\phi = 4\pi A - 2\pi\,(-3E_0) \int_{-1}^{1} \cos\theta \, d(\cos\theta) = Q_0/\epsilon_0 \tag{52}$$

$$\Rightarrow A = \frac{Q_0}{4\pi\epsilon_0} \tag{53}$$

Thus the final expression for $\vec{E}$ is

$$\boxed{\vec{E} = \left[\frac{Q_0}{4\pi\epsilon_0 r^2} + E_0\left(1 + \frac{2R^3}{r^3}\right)\cos\theta\right]\hat{\mathbf{r}} - E_0 \sin\theta \left(1 - \frac{R^3}{r^3}\right)\hat{\boldsymbol{\theta}}} \tag{54}$$

(b) The radial component E_r of the field at $r = R$ is

$$E_r(R) = \frac{Q_0}{4\pi\epsilon_0 R^2} + 3E_0\cos\theta \tag{55}$$

The field has a radially-inward component where $E_r(R) < 0$, which occurs for

$$\cos\theta < -\frac{Q_0}{12\pi\epsilon_0 E_0 R^2} \equiv -\frac{Q_0}{Q_{\max}}. \tag{56}$$

Let's check limiting cases. We know $-1 \leq \cos\theta \leq 1$. If $Q_0 > Q_{\max}$, the field will never have a radially-inward component. If $Q_0 < -Q_{\max}$, the field will have a radially-inward component everywhere on the surface. If $-Q_{\max} < Q < Q_{\max}$, the field will only have a radially-inward component in the θ-range satisfying $\cos\theta < -Q_0/Q_{\max}$.

(c) We found in part **(b)** that $\vec{E}$ will only have a radially-inward component if $Q_0 < Q_{\max} = 12E_0\pi\epsilon_0 R^2$. If $Q_0 > Q_{\max}$, the field will never have an inward component at the surface and the positively-charged dust will be repelled by the surface, never settling onto the surface.

(d) The radial component of the current density J_r is given by

$$\boxed{J_r = \rho_0\mu E_r(R) = 3\rho_0\mu E_0\left(\frac{Q}{Q_{\max}} + \cos\theta\right)} \tag{57}$$

Notice that the current density vanishes if $\cos\theta = -Q_0/Q_{\text{max}}$, as expected.

(e) The rate at which charge settles onto the sphere has two cases:

Case 1: $Q(t) < -Q_{\text{max}}$
In this case, $E_r < 0$ everywhere on the sphere, so dust can settle onto the entire surface of the sphere.

$$\frac{dQ}{dt} = -\int_0^{2\pi} d\phi \int_{-1}^{1} J_r R^2 d(\cos\theta) \tag{58}$$

$$= -6\pi\rho_0\mu E_0 R^2 \int_{-1}^{1} \left(\frac{Q}{Q_{\text{max}}} + \cos\theta\right) d(\cos\theta) \tag{59}$$

$$= -6\pi\rho_0\mu E_0 R^2 \times \frac{2Q}{Q_{\text{max}}} \tag{60}$$

Normalizing to Q_{max}, we have

$$\boxed{\frac{1}{Q_{\text{max}}}\frac{dQ}{dt} = -\frac{\rho_0\mu}{\epsilon_0}\frac{Q}{Q_{\text{max}}}, \quad Q(t) < -Q_{\text{max}}} \tag{61}$$

Case 2: $-Q_{\text{max}} < Q(t) < Q_{\text{max}}$
In this case, $E_r(R) < 0$ only for $\theta_c < \theta < \pi$, where $\theta_c = \arccos[-Q/Q_{\text{max}}]$. The integral is the same as before, only with a restricted θ-range:

$$\frac{dQ}{dt} = -6\pi\rho_0\mu E_0 R^2 \int_{-1}^{\cos\theta_c} \left(\frac{Q}{Q_{\text{max}}} + \cos\theta\right) d(\cos\theta) \tag{62}$$

$$= -6\pi\rho_0\mu E_0 R^2 \left[\left(\frac{Q}{Q_{\text{max}}}\right)(1 + \cos\theta_c) + \frac{1}{2}\left(\cos^2\theta_c - 1\right)\right] \tag{63}$$

Normalizing to Q_{max}, and doing a little simplification using $\cos\theta = -Q/Q_{\text{max}}$

$$\boxed{\frac{1}{Q_{\text{max}}}\frac{dQ}{dt} = -\frac{\rho\mu}{4\epsilon_0}\left(1 - \frac{Q}{Q_{\text{max}}}\right), \quad -Q_{\text{max}} < Q(t) < Q_{\text{max}}} \tag{64}$$

Quantum Mechanics Problem 1

(a) The matrix representation of the $\hat{S}_{\vec{n}}$ operator is

$$\hat{S}_{\vec{n}} \doteq \frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix} \tag{65}$$

The eigenvalues of this matrix are given by

$$\det\begin{pmatrix}\cos\theta-\lambda & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta-\lambda\end{pmatrix} = \mathbf{0} \tag{66}$$

resulting in the characteristic equation

$$-\cos^2\theta + \lambda^2 - \sin^2\theta = 0 \tag{67}$$

$$\Rightarrow \lambda = \pm 1 \tag{68}$$

Thus, the eigenvalues of $\hat{S}_{\vec{n}}$ are $\boxed{\pm\hbar/2}$. The eigenvectors $|\chi_\pm\rangle$ satisfy $\hat{S}_{\vec{n}}\,|\chi_\pm\rangle = \pm\frac{\hbar}{2}\,|\chi_\pm\rangle$

Case 1: $\lambda = \hbar/2$ (χ_+)

$$\begin{pmatrix}\cos\theta-1 & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta-1\end{pmatrix}\begin{pmatrix}\chi_1 \\ \chi_2\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix} \tag{69}$$

$$\begin{cases}\chi_1(\cos\theta-1) + \chi_2\sin\theta e^{-i\phi} = 0 \\ \chi_1\sin\theta e^{i\phi} - \chi_2(\cos\theta+1) = 0\end{cases} \tag{70}$$

To simplify this, use the identities $\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2)$, $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$, and $\cos^2(\theta/2) + \sin^2(\theta/2) = 1$:

$$\begin{cases}-2\chi_1\sin^2(\theta/2) + 2\chi_2\sin(\theta/2)\cos(\theta/2)e^{-i\phi} = 0 \\ 2\chi_1\sin(\theta/2)\cos(\theta/2)e^{i\phi} - 2\chi_2\cos^2(\theta/2) = 0\end{cases} \tag{71}$$

This implies

$$\frac{\chi_1}{\chi_2} = \frac{\cos(\theta/2)}{\sin(\theta/2)}e^{-i\phi} \tag{72}$$

from which we can read off the (already normalized!) eigenvector

$$\boxed{\chi_+ = \begin{pmatrix}\cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2}\end{pmatrix}} \tag{73}$$

The overall phase of the eigenvector is undetermined; however, it's conventional to split the $e^{i\phi}$ term into both components of χ to make the θ and ϕ components look more symmetric.

Case 2: $\lambda = -\hbar/2$ (χ_-)
The only difference here is that the eigenvector constraint equations have slightly different signs:

$$\begin{cases}\chi_1(\cos\theta+1) + \chi_2\sin\theta e^{-i\phi} = 0 \\ \chi_1\sin\theta e^{i\phi} - \chi_2(\cos\theta-1) = 0\end{cases} \tag{74}$$

Everything proceeds exactly the same, and we end up with the eigenvector

$$\boxed{\chi_- = \begin{pmatrix}-\sin(\theta/2)e^{-i\phi/2} \\ \cos(\theta/2)e^{i\phi/2}\end{pmatrix}}. \tag{75}$$

(b) Without loss of generality, let $|\psi\rangle = |\chi_+\rangle$ (this is equivalent to choosing a coordinate system such that $\vec{n}$ points in the direction of χ_+). From $\langle\psi|\,\hat{S}_z\,|\psi\rangle = 0$, we obtain

$$\langle\psi|\,\hat{S}_z\,|\psi\rangle = \frac{\hbar}{2}\left[\cos^2(\theta/2) - \sin^2(\theta/2)\right] = 0 \tag{76}$$

$$\Rightarrow \sin(\theta/2) = \pm\cos(\theta/2) \tag{77}$$

Choose the (+) solution (the (-) solution works just as well). The corresponding state is

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\phi/2} \\ e^{i\phi/2} \end{pmatrix} \tag{78}$$

Using the $\hat{S}_x$ and $\hat{S}_y$ provided, we can calculate

$$\langle\psi| \hat{S}_x |\psi\rangle \propto e^{-i\phi} + e^{i\phi}$$
$$\langle\psi| \hat{S}_y |\psi\rangle \propto e^{-i\phi} - e^{i\phi}$$

It is impossible for both $\langle\psi| \hat{S}_x |\psi\rangle$ and $\langle\psi| \hat{S}_y |\psi\rangle$ to be zero if $\langle\psi| \hat{S}_x |\psi\rangle = 0$.

(c) The unitary operator $U(t)$ describing the time-evolution of the quantum system is

$$\hat{U}(t) = e^{-i\hat{H}(t)/\hbar} \tag{79}$$

with the Hamiltonian

$$\frac{\hat{H}}{\hbar} = -\frac{g\mu B}{2}\hat{\sigma}_y \equiv -\gamma\hat{\sigma}_y \tag{80}$$

Thus the time-evolution operator is

$$\hat{U}(t) = e^{i\gamma t\hat{\sigma}_y} = \cos(\gamma t) + i\hat{\sigma}_y \sin(\gamma t) \tag{81}$$

$$\hat{U}(t) \doteq \begin{pmatrix} \cos(\gamma t) & \sin(\gamma t) \\ -\sin(\gamma t) & \cos(\gamma t) \end{pmatrix} \tag{82}$$

For an initial state

$$|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{83}$$

we have a state after time T

$$|\psi(T)\rangle = \begin{pmatrix} \cos(\gamma T) \\ -\sin(\gamma T) \end{pmatrix} \tag{84}$$

The eigenvector of $\hat{S}_x$ with eigenvalue $+\hbar/2$ is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{85}$$

so the probability of measuring this state after time T has elapsed is

$$\boxed{\text{Prob}(\langle\hat{S}_x\rangle = +\hbar/2) = \frac{1}{2}\left[\cos(\gamma T) - \sin(\gamma T)\right]^2} \tag{86}$$

Quantum Mechanics Problem 2

(a) The translation operator $\hat{T}(d)$ translates a state by a distance d:

$$\hat{T}(d) = e^{i\hat{p}d/\hbar}. \tag{87}$$

The probability of finding the oscillator in state $|n\rangle$ after translating the ground state $|0\rangle$ by a distance d is $\text{Prob}(n) = |\langle n|\,\hat{T}(d)\,|0\rangle|^2$. To calculate $\hat{T}(d)$, we need to express $\hat{p}$ in terms of the creation/annihilation operators:

$$\hat{a} - \hat{a}^\dagger = \sqrt{\frac{1}{2\hbar\omega}}(i\hat{p} + i\hat{p}) \quad \Rightarrow \quad i\hat{p} = \sqrt{\frac{\hbar\omega}{2}}(\hat{a} - \hat{a}^\dagger) \tag{88}$$

We thus find

$$\hat{T}(d) = e^{i\hat{p}d/\hbar} = \exp\left[d\sqrt{\frac{\omega}{2\hbar}}(\hat{a} - \hat{a}^\dagger)\right] \equiv \exp\left[r(\hat{a} - \hat{a}^\dagger)\right] \tag{89}$$

We are given the operator identity

$$e^{(\hat{A}+\hat{B})} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]} \tag{90}$$

which is valid if $[\hat{A},\hat{B}]$ is proportional to a unit operator. The creation operators are defined such that $[\hat{a},\hat{a}^\dagger] = 1$, and we can make the identifications $\hat{A} = -r\hat{a}^\dagger$ and $\hat{B} = r\hat{a}$ which gives

$$e^{r(\hat{a}-\hat{a}^\dagger)} = e^{-r\hat{a}^\dagger}e^{r\hat{a}}e^{-r^2/2}. \tag{91}$$

Thus

$$\langle n|\,\hat{T}(d)\,|0\rangle = e^{-r^2/2}\,\langle n|\,e^{-r\hat{a}^\dagger}e^{r\hat{a}}\,|0\rangle \tag{92}$$

Expanding $e^{r\hat{a}} = 1 + r\hat{a} + \ldots$ and noting that $\hat{a}\,|0\rangle = 0$ (so only the leading term survives), we have

$$\begin{aligned}\langle n|\,\hat{T}(d)\,|0\rangle &= e^{-r^2/2}\,\langle n|\,e^{-r\hat{a}^\dagger}\,|0\rangle & (93)\\ &= e^{-r^2/2}\sum_{\ell=0}^{\infty}\left\langle n\left|\frac{(-1)^\ell r^\ell(\hat{a}^\dagger)^\ell}{\ell!}\right|0\right\rangle & (94)\end{aligned}$$

Since we also have

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\,|0\rangle\,, \tag{95}$$

the only non-vanishing term in the sum is $\ell = n$, from which we find

$$\langle n|\,\hat{T}(d)\,|0\rangle = e^{-r^2/2}\frac{(-r)^n}{\sqrt{n!}} \tag{96}$$

and thus

$$\boxed{\text{Prob}(n) = e^{-r^2}\frac{r^{2n}}{n!}} \tag{97}$$

(b) Let the wavefunction at $t = 0$ be denoted $|\psi(0)\rangle$. The time evolution of the state under the Hamiltonian $\hat{H} = \left(n + \frac{1}{2}\right)\hbar\omega$ is given by

$$\begin{aligned}|\psi(t)\rangle &= e^{-i\hat{H}t/\hbar}\,|\psi(0)\rangle & (98)\\ &= \sum_{n=0}^{\infty} e^{-i\omega t\left(n+\frac{1}{2}\right)}\,|n\rangle\,\langle n|\psi(0)\rangle & (99)\end{aligned}$$

where I have inserted a complete set of states. The amplitude is periodic in time with period $2\pi/\omega$; hence, the probabilities of obtaining a given momentum will be the same at $t = 0,\ 2\pi/\omega,\ 4\pi/\omega,\ \ldots$

(c) The probability of measuring a given energy is independent of t, since the eigenstates of $\hat{H}$ are *stationary states*.

Statistical Mechanics Problem 1

(a) From the first law of thermodynamics, $dU = dQ - dW$, where dQ is the heat absorbed *by* the gas and dW is the work done *by* the gas.

Steps 1 and 3: isothermal. Since the ideal gas has $U \propto T$, an isothermal process ($T = \text{const}$) has $dU = 0$. Thus

$$dQ = dW = PdV = \frac{NkT}{V}\, dV \tag{100}$$

The magnitude of the heat exchanged with the T_{H} reservoir is

$$|Q_{\mathrm{H}}| = NkT_{\mathrm{H}} \ln\left(\frac{V_b}{V_a}\right) \quad (V_b > V_a) \tag{101}$$

Similarly, the magnitude of the heat exchanged with the cold reservoir is

$$|Q_{\mathrm{L}}| = NkT_{\mathrm{L}} \ln\left(\frac{V_c}{V_d}\right) \quad (V_c > V_d) \tag{102}$$

Steps 2 and 4: adiabatic. Since adiabatic processes have $dQ = 0$ by definition, $dU = -dW$ and therefore (using $U = \frac{f}{2}NkT$ for an ideal gas with f degrees of freedom per particle)

$$\frac{fNk}{2}\, dT = -PdV \tag{103}$$

$$\frac{f}{2}(PdV + VdP) = -PdV \tag{104}$$

$$\left(\frac{f+2}{2}\right) PdV = \frac{f}{2} VdP \tag{105}$$

$$\frac{dP}{P} = \left(\frac{f+2}{f}\right)\frac{dV}{V} \equiv \gamma \frac{dV}{V} \tag{106}$$

We thus find

$$\left(\frac{P_c}{P_b}\right) = \left(\frac{V_c}{V_b}\right)^{\gamma} \quad \text{and} \quad \left(\frac{P_a}{P_d}\right) = \left(\frac{V_a}{V_d}\right)^{\gamma} \tag{107}$$

The internal energy can also be written

$$U = \frac{f}{2}NkT = \frac{f}{2}PV \tag{108}$$

for an ideal gas. Since the internal energy is constant along the isotherms $a \to b$ and $c \to d$,

$$P_a V_a = P_b V_b \quad \text{and} \quad P_c V_c = P_d V_d \tag{109}$$

Combining these with the adiabatic results, we have

$$\frac{V_b V_d}{V_a V_c} = \left(\frac{V_a V_c}{V_b V_d}\right)^{\gamma} \quad \Rightarrow \quad \frac{V_b}{V_a} = \frac{V_c}{V_d} \tag{110}$$

Since the efficiency e is given by

$$e = 1 - \frac{|Q_{\mathrm{L}}|}{|Q_{\mathrm{H}}|}, \tag{111}$$

we can substitute $V_b/V_a = V_c/V_d$ into the expressions for $|Q_{\mathrm{L}}|$ and $|Q_{\mathrm{H}}|$ to obtain

$$\boxed{e = 1 - \frac{T_{\mathrm{L}}}{T_{\mathrm{H}}}} \tag{112}$$

(b) The heat capacity at constant pressure C_P is given by (in the limit that $\Delta T \to 0$)

$$C_P = \left(\frac{\Delta Q}{\Delta T}\right)_P = \left(\frac{\Delta U + \Delta W}{\Delta T}\right)_P = \left(\frac{\partial U}{\partial T}\right)_P + P\left(\frac{\partial V}{\partial T}\right)_P \tag{113}$$

Using $U = \frac{f}{2}NkT$ and $PV = NkT$, we have

$$\boxed{C_P = \frac{f}{2}Nk + Nk = Nk\left(1 + \frac{f}{2}\right)} \tag{114}$$

Similarly, we have

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V \tag{115}$$

(the contribution from $dW = PdV$ drops out because we're constrained to $dV = 0$). Thus

$$\boxed{C_V = \frac{f}{2}Nk} \tag{116}$$

As a sanity check, we can calculate

$$\frac{C_P}{C_V} = \frac{f+2}{f} = \gamma, \tag{117}$$

where γ is exactly the adiabatic index from part **(a)**, as expected.

Statistical Mechanics Problem 2

(a) The number of electrons with $k < k_{\mathrm{F}}$ is given by

$$N = 2\int d^3x \int_{k<k_{\mathrm{F}}} \frac{d^3k}{(2\pi)^3} \tag{118}$$

(the initial factor of 2 accounts for the two electron spin states). Since the volume is just V, and the angular part of the k integral gives 4π (no angular dependence in the integrand) we have

$$N = 2V \times \frac{4\pi}{8\pi^3}\int_0^{k_{\mathrm{F}}} k^2\, dk \tag{119}$$

$$\boxed{k_{\mathrm{F}} = \left(\frac{3\pi^2 N}{V}\right)^{1/3} = \left(3\pi^2 n\right)^{1/3}} \tag{120}$$

The energy of an individual electron with momentum k is just $\epsilon = \hbar^2k^2/2m$, so to find the total energy E_0 of the system in its ground state, we integrate ϵ against d^3k:

$$E_0 = 2V \times \frac{4\pi}{8\pi^3}\int_0^{k_{\mathrm{F}}} \frac{\hbar^2 k^4}{2m} dk = \frac{\hbar^2 V k_{\mathrm{F}}^5}{10\pi^2 m} \tag{121}$$

Thus

$$\frac{E_0}{V} = \frac{\hbar^2 k_{\mathrm{F}}^5}{10\pi^2 m} \tag{122}$$

$$= \frac{\hbar^2}{10\pi^2 m}\left(3\pi^2 n\right)^{5/3} \tag{123}$$

$$\boxed{\frac{E_0}{V} = \frac{3}{5} \times \frac{\hbar^2}{2m} \times \left(3\pi^2\right)^{2/3} n^{5/3}} \tag{124}$$

(b) The numbers of electrons with spin up (+) or down (-), $N_\pm$ is calculated the same way as in part **(a)**, except without the factor of 2 (since we're now considering each spin orientation separately):

$$N_\pm = V \times \frac{4\pi}{8\pi^3}\int_0^{k_{\mathrm{F},\pm}} k^2\, dk \tag{125}$$

$$\boxed{k_{\mathrm{F},\pm} = \left(6\pi^2 n_\pm\right)^{1/3}} \tag{126}$$

(c) The kinetic energy of the system is given by adding the energy of the two spin polarizations

$$\frac{E_{\mathrm{kin}}}{V} = V \times \frac{4\pi}{8\pi^3}\left[\int_0^{k_{\mathrm{F},+}} \frac{\hbar^2 k_+^4}{2m}\, dk_+ + \int_0^{k_{\mathrm{F},-}} \frac{\hbar^2 k_-^4}{2m}\, dk_-\right] \tag{127}$$

$$= \frac{1}{10\pi^2}\frac{\hbar^2}{2m}\left(k_{\mathrm{F},+}^5 + k_{\mathrm{F},-}^5\right) \tag{128}$$

which reduces to the result from part **(a)** if $N_+ = N_- = N/2$. In terms of the number density n, we have

$$\frac{E_{\mathrm{kin}}}{V} = \frac{1}{10\pi^2} \times \frac{\hbar^2}{2m} \times (6\pi^2)^{5/3}\left(n_+^{5/3} + n_-^{5/3}\right) \tag{129}$$

Setting $n_\pm = \frac{n}{2} \pm \delta$, we have

$$\frac{E_{\mathrm{kin}}}{V} = \frac{1}{10\pi^2} \times \frac{\hbar^2}{2m} \times \left(6\pi^2\right)^{5/3} \times \left(\frac{n}{2}\right)^{5/3}\left[\left(1 + \frac{2\delta}{n}\right)^{5/3} + \left(1 - \frac{2\delta}{n}\right)^{5/3}\right] \tag{130}$$

$$= \frac{1}{10\pi^2} \times \frac{\hbar^2}{2m} \times \left(3\pi^2 n\right)^{5/3}\left[2 + \frac{5}{3}\left(\frac{5}{3} - 1\right)\left(\frac{2\delta}{n}\right)^2 + \mathcal{O}(\delta^4)\right] \tag{131}$$

$$= \frac{3}{5} \times \frac{\hbar^2}{2m} \times \left(3\pi^2\right)^{2/3} \times n^{5/3} + \frac{4}{3} \times \frac{\hbar^2 n^{-1/3}}{2m} \times \left(3\pi^2\right)^{2/3} \tag{132}$$

$$\boxed{\frac{E_{\mathrm{kin}}}{V} \approx E_0 + \frac{4}{3}\left(3\pi^2\right)^{2/3} \frac{\hbar^2 n^{-1/3}}{2m} \delta^2} \tag{133}$$

(d) From the definition,

$$\boxed{\frac{U}{V} = \alpha n_+ n_- = \alpha\left(\frac{n^2}{4} - \delta^2\right)} \tag{134}$$

(e) Including our new U/V term in the total energy, we have

$$\frac{E}{V} \approx E_0 + \frac{\alpha n^2}{4} + \left[\frac{4}{3}\left(3\pi^2\right)^{2/3} \frac{\hbar^2 n^{-1/3}}{2m} - \alpha\right] \delta^2 \tag{135}$$

The system can lower its total energy by developing magnetization if $[\ldots] < 0$, which occurs at the critical value

$$\boxed{\alpha_c = \frac{4}{3}\left(3\pi^2\right)^{2/3} \frac{\hbar^2 n^{-1/3}}{2m}} \tag{136}$$