MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS

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DOCTORAL GENERAL EXAMINATION

PART II

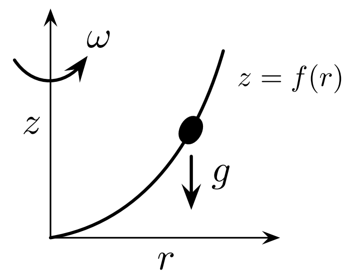
Friday, February 3, 2012
9:30 a.m. - 2:30 p.m., Room 32-082

FIVE HOURS

1. This examination is divided into four sections, Mechanics, Electricity & Magnetism, Statistical Mechanics, and Quantum Mechanics, with two problems in each. Read both problems in each section carefully before making your choice. Submit ONLY one problem per section. IF YOU SUBMIT MORE THAN ONE PROBLEM FROM A SECTION, BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.
2. For each problem, use the separate booklet that you have been given. Do not put your name on it, as each booklet has an identification number that will allow the papers to be graded blindly. Please, however, write the problem number (I.2 for example) on the front of each booklet.
3. Calculators may not be used.
4. No books or reference materials may be used.

SECTION I: CLASSICAL MECHANICS

Classical Mechanics 1: Bead on a Curved Wire



A bead of mass m slides without friction along a curved wire with shape $z = f(r)$, as indicated in the figure above, with $r = \sqrt{x^2 + y^2}$. The wire is rotated around the z-axis at a constant angular velocity ω, keeping its shape fixed. Gravity acts downward, with an acceleration $g > 0$.

(a) (*2 pts*) Using Newton's second law ($\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$) in an inertial frame, derive an expression for the radius r_0 of a fixed circular orbit (i.e. a solution with $r = r_0 = \text{const.}$). What is the normal force the wire applies to the bead to keep it in a circular orbit?

(b) (*2 pts*) Write down the one-dimensional Lagrangian $L(r, \dot{r}, t)$ for this system. Using this Lagrangian, obtain an equation of motion for $r(t)$ and verify your result for r_0.

(c) (*2 pts*) Now consider a small displacement from the circular orbit, $r = r_0 + \epsilon(t)$. Derive a condition on the function $f(r)$ such that a circular orbit at $r = r_0$ is stable.

(d) (*2 pts*) Find the component of the force on the bead in the $\hat{\boldsymbol{e}}_\phi$ direction, i.e. perpendicular to the plane of the wire. (The angular velocity is $\omega = d\phi/dt$.) This force is sometimes called the "constraint force." Obtain an answer that is valid for arbitrary motion of the bead; i.e., do not assume that $r = r_0$ or $r = r_0 + \epsilon(t)$. [Hint: you can solve this problem either by using $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$, or by using Lagrange multipliers. If you use Lagrange multipliers, remember that they can only impose (holonomic) constraints on coordinates, and constraints directly on velocities are more subtle.]

(e) (*2 pts*) Find the Hamiltonian $H(r, p_r, t)$ and show that it is conserved. By what amount does H differ from the total energy of the bead? Is the total energy automatically conserved? Explain why or why not in terms of your answer to part (d).

SOLUTION:

(a) The centripetal acceleration needed to keep the bead in a circular orbit is $\vec{\boldsymbol{a}} = -\omega^2 r_0 \hat{\boldsymbol{r}}$, where $\hat{\boldsymbol{r}}$ is the unit vector pointing along the radial direction. Newton's law thus becomes

$$\vec{\boldsymbol{F}}_N - mg\hat{\boldsymbol{z}} = -m\omega^2 r_0 \hat{\boldsymbol{r}}, \tag{1}$$

where $\vec{\boldsymbol{F}}_N$ is the normal force applied by the wire. Requiring that the force be normal gives

$$\vec{\boldsymbol{F}}_N = -F_N \cos\phi \hat{\boldsymbol{r}} + F_N \sin\phi \hat{\boldsymbol{z}}, \qquad \cot\phi = f'(r), \tag{2}$$

where ϕ is the angle the wire makes to the vertical. The two components of Newton's law are

$$\hat{\boldsymbol{r}} : -F_N \cos\phi = -m\omega^2 r_0, \qquad \hat{\boldsymbol{z}} : F_N \sin\phi - mg = 0. \tag{3}$$

Solving for r_0 and $\vec{\boldsymbol{F}}_N$, we obtain

$$\frac{f'(r_0)}{r_0} = \frac{\omega^2}{g}, \qquad \vec{\boldsymbol{F}}_N = mg\left(-f'(r_0)\hat{\boldsymbol{r}} + \hat{\boldsymbol{z}}\right) = -m\omega^2 r_0 \hat{\boldsymbol{r}} + mg\hat{\boldsymbol{z}}. \tag{4}$$

(b) For an unconstrained system rotating with angular frequency ω and subject to a gravitational potential, the Lagrangian would be

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\omega^2 r^2 + \frac{1}{2}m\dot{z}^2 - mgz. \tag{5}$$

Since y is constrained to be $y = f(r)$, the Lagrangian is

$$L = \frac{1}{2}m\dot{r}^2(1 + f'(r)^2) + \frac{1}{2}m\omega^2 r^2 - mgf(r). \tag{6}$$

The Euler-Lagrange equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$ yields

$$\ddot{r}\left(1 + f'(r)^2\right) + \dot{r}^2 f'(r)f''(r) + gf'(r) - \omega^2 r = 0. \tag{7}$$

To have a solution with $\dot{r} = \ddot{r} = 0$, we require

$$gf'(r_0) - \omega^2 r_0 = 0, \tag{8}$$

in agreement with Eq. (4).

(c) Expanding to first order in $r(t) = r_0 + \epsilon(t)$, the equation of motion becomes

$$\ddot{\epsilon}\left(1 + (f_0')^2\right) + (gf_0'' - \omega^2)\epsilon = 0, \tag{9}$$

where we have used $f'(r) = f'(r_0) + \epsilon(t)f''(r_0) \equiv f_0' + f_0''\epsilon(t)$. To have a stable orbit requires

$$f_0'' > \frac{\omega^2}{g}, \tag{10}$$

corresponding to a positive effective "spring constant".

(d) The constraint force can be obtained by promoting the angular variable ϕ to a coordinate in the Lagrangian, and imposing the constraint $\phi = \omega t$ through a Lagrange multiplier λ. (It is a bit more subtle to introduce the constraint $\dot\phi = \omega$, though that is possible as well.) The new Lagrangian is

$$L = \frac{1}{2}m\dot r^2(1 + f'(r)^2) + \frac{1}{2}m\dot\phi^2 r^2 - mgf(r) + \lambda(\phi - \omega t), \tag{11}$$

and the Euler-Lagrange equation for ϕ, after making the replacements $\dot\phi = \omega$ and $\ddot\phi = 0$, becomes

$$2mr\dot r\omega = \lambda\ . \tag{12}$$

Since the tangential distance element is $rd\phi$, the tangential constraint force is

$$F_\phi = \frac{1}{r}\frac{\partial L}{\partial \phi} = \frac{\lambda}{r} = 2m\omega\dot r. \tag{13}$$

An alternative solution is to use $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ directly. The position vector is

$$\vec{\boldsymbol{r}} = r\hat{\boldsymbol{r}} + f(r)\hat{\boldsymbol{z}}, \tag{14}$$

and using the fact that $d\hat{\boldsymbol{r}}/dt = \omega\hat{\boldsymbol{\phi}}$, the velocity vector is

$$\vec{\boldsymbol{v}} = \dot r\hat{\boldsymbol{r}} + \omega r\hat{\boldsymbol{\phi}} + \dot r f'(r)\hat{\boldsymbol{z}}. \tag{15}$$

The constraint force is the only force in the $\hat{\boldsymbol{\phi}}$ direction, and the component of the acceleration $\vec{\boldsymbol{a}} = d\vec{\boldsymbol{v}}/dt$ along the $\hat{\boldsymbol{\phi}}$ direction is

$$a_\phi = 2\omega\dot r. \tag{16}$$

Using Newton's second law, we recover

$$F_\phi = 2m\omega\dot r, \tag{17}$$

in agreement with Eq. (13).

(e) The canonical momentum is

$$p_r = \frac{\partial L}{\partial \dot r} = m\dot r(1 + f'(r)^2), \tag{18}$$

and the Hamiltonian is thus

$$H \equiv p_r\dot r - L = \frac{p_r^2}{2m}\frac{1}{1 + f'(r)^2} - \frac{1}{2}m\omega^2 r^2 + mgf(r). \tag{19}$$

Since the Hamiltonian does not depend explicitly on time, it is conserved. The total energy (kinetic plus potential) is

$$E = T + V = H + m\omega^2 r^2, \tag{20}$$

so it differs from the Hamiltonian by the centripetal term. The total energy is only conserved if $\dot{r} = 0$, since if $\dot{r} \neq 0$, then the wire applies a constraint force on the bead in the direction of motion. (The other components of the constraint force do no work since they are perpendicular to the direction of motion). In particular, the motor that keeps the keeps the wire rotating at a constant speed can do work on the system. Though not necessary for full credit, from part (d) and using the constraint $\phi = \omega t$, the work done by the wire is

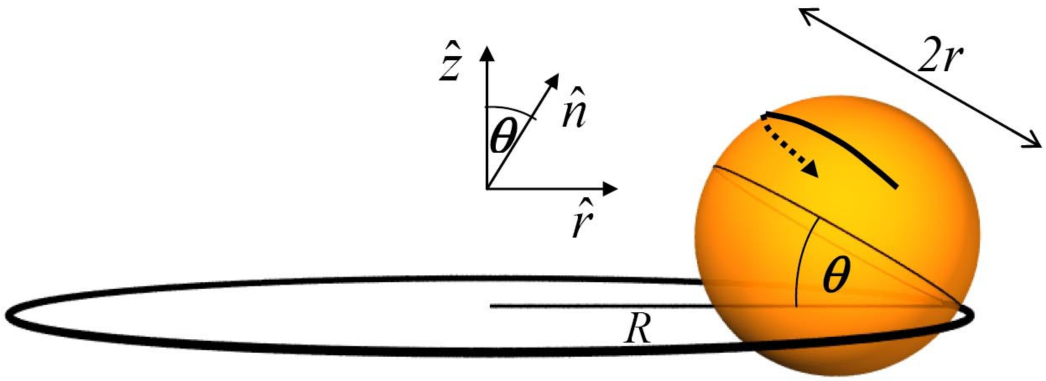
$$dW = F_\phi (r d\phi) = 2m\dot{r}\omega r d\phi = m\frac{\partial r^2}{dt}\omega^2 dt, \tag{21}$$

which can be integrated up to give

$$W = m\omega^2 r^2. \tag{22}$$

As expected, the total energy exceeds the Hamiltonian by the work done on the bead by the constraint force.

Classical Mechanics 2: Basketball on a Rim



A basketball of radius r rolls without slipping around a basketball rim of radius R. The basketball rotates in such a way that the contact point traces out a great circle on the ball (i.e. a circle with maximum circumference $2\pi r$), and the center of mass moves in a horizontal circle with angular frequency Ω, counterclockwise as seen from above. The plane of the great circle makes an angle θ with the horizontal. The ball has a moment of inertia $I = \frac{2}{3}mr^2$ around its center, and gravity acts downward with an acceleration of magnitude $g > 0$.

(a) (*3 pts*) Calculate the torque on the ball about its center of mass imparted by gravity and the contact force from the hoop.

(b) (*3 pts*) Determine the angular velocity vector $\vec{\omega}$ that describes the rotation of the ball relative to the inertial frame. Express your answer in terms of Ω, R, r, and suitable unit vectors. [*Hint:* It may be helpful to consider the rotating frame in which the center of mass is at rest, but be sure to give your answer in the original frame.]

(c) (*4 pts*) Find Ω in terms of g, R, r, and θ.

SOLUTION:

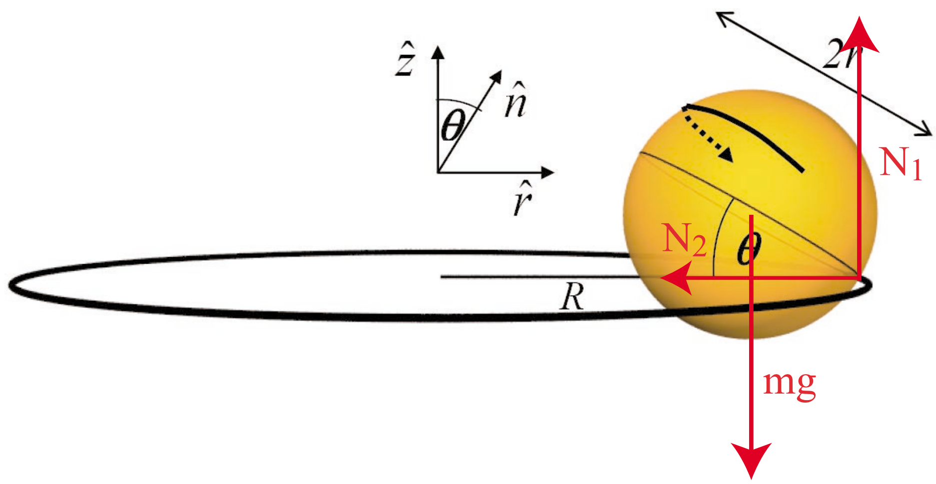


Figure 1: Forces acting on the Basketball

(a) The forces are gravity, acting on the center of mass, and the contact force that the rim exerts on the basketball. The contact force consists of two parts, one vertical, N_1 to compensate against gravity, the other, N_2, horizontal, along $-\hat{r}$, to provide the necessary centripetal acceleration of the basketball as its center of mass moves around the circle with angular frequency Ω. Note that the radius of the circle traced out by the center of mass is $R - r\cos\theta$. We thus have for the magnitudes of the forces:

$$N_1 = mg$$

and

$$N_2 = m\Omega^2(R - r\cos\theta).$$

Next, we consider the torque about the center of mass. Gravity acts on the center of mass and thus does not exert a torque. However, the two contact forces N_1 and N_2 do exert torques. N_1, the vertical normal force, results in a torque τ_1 directed out of the paper, in the direction opposite the basketball's center of mass velocity, while N_2 results in a torque τ_2 into the paper, aligned with the basketball's CM velocity. We have $\tau_1 = N_1 r\cos\theta$ and $\tau_2 = N_2 r\sin\theta$. For the motion to occur as described, we need $\tau_1 > \tau_2$, as the spin angular momentum of the basketball has to keep pointing inwards (see Fig. 2). The total torque is

$$\boxed{\tau = \tau_1 - \tau_2 = r(N_1\cos\theta - N_2\sin\theta) = mgr\cos\theta - m\Omega^2 r(R - r\cos\theta)\sin\theta\ ,} \tag{1}$$

directed out of the page.

(b) In the rotating frame in which the center of mass is at rest, the angular velocity vector $\vec{\boldsymbol{\omega}}_s$ of the basketball lies in the direction $-\hat{\boldsymbol{n}}$. We can obtain ω_s from the non-slipping

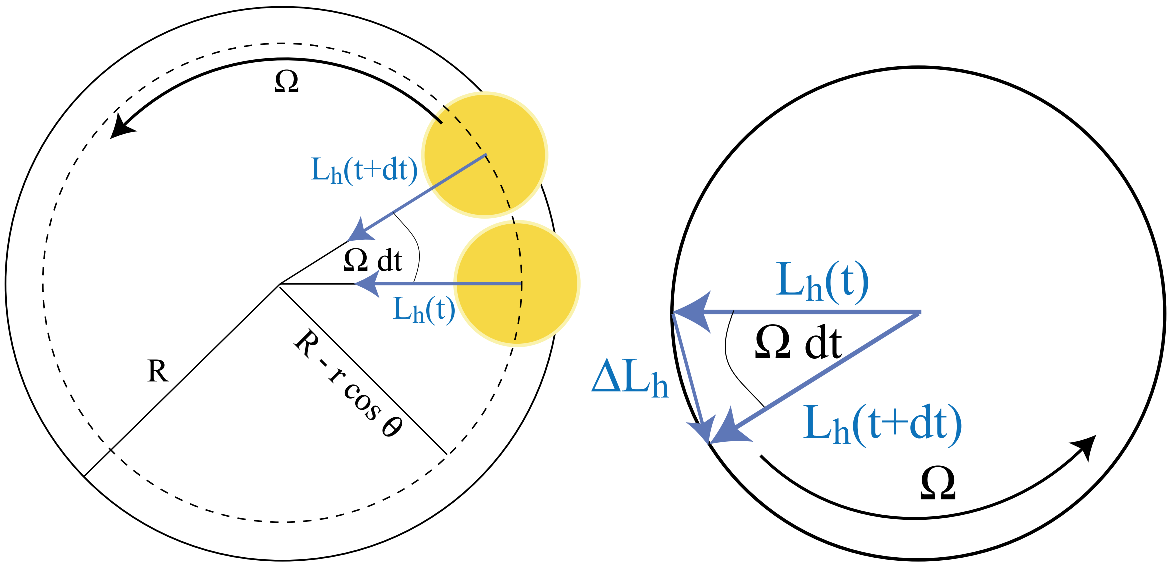


Figure 2: Precession of basketball spin angular momentum

condition for the basketball's rotation along the rim. As the basketball rotates by an angle $\omega_s dt$ about its own axis, the point on the basketball that was initially in contact with the rim will have traveled a distance $\omega_s dt r$ on the grand circle on its surface. At the same time, the contact point between basketball and rim will have traveled the equal distance $\Omega dt R$. Thus $\omega_s = \frac{R}{r}\Omega$. The angular velocity vector in the rotating frame is thus $\vec{\boldsymbol{\omega}}_s = -\frac{R}{r}\Omega\hat{\boldsymbol{n}}$. To go back to the "lab" frame, we add $\Omega\hat{z}$ to find

$$\boxed{\vec{\boldsymbol{\omega}} = \Omega\hat{z} - \frac{R}{r}\Omega\hat{\boldsymbol{n}}\ .} \tag{2}$$

(c) Since the basketball is spherically symmetric, the moment of inertia tensor is simply the unit matrix times $I = \frac{2}{3}mr^2$. The angular momentum about the center of mass is then $\vec{\boldsymbol{L}} = I\vec{\boldsymbol{\omega}}$ and thus

$$\vec{\boldsymbol{L}} = I\Omega\left(\hat{z} - \frac{R}{r}\hat{\boldsymbol{n}}\right)\ . \tag{3}$$

Only the horizontal part of the angular momentum has to change as the basketball rolls around the rim. This part has magnitude

$$L_h = I\Omega\frac{R}{r}\sin\theta = \frac{2}{3}m\Omega Rr\sin\theta\ . \tag{4}$$

We know that $\vec{\boldsymbol{L}}_h$ has to rotate about the z-axis with frequency Ω, to track the basketball's motion. For a small time-step Δt we thus see that

$$\Delta L_h = L_h\Omega\Delta t \tag{5}$$

or

$$\left|\frac{d\vec{\boldsymbol{L}}_h}{dt}\right| = L_h\Omega = \frac{2}{3}m\Omega^2 Rr\sin\theta \ . \tag{6}$$

This change of L_h has to be given by the torque, which is indeed orthogonal to L_h.

$$\left|\frac{d\vec{\boldsymbol{L}}_h}{dt}\right| = \tau \tag{7}$$

or

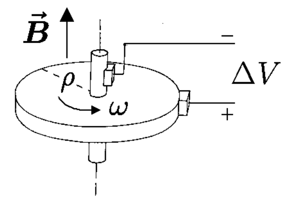
$$\frac{2}{3}m\Omega^2 Rr\sin\theta = mgr\cos\theta - m\Omega^2 r(R - r\cos\theta)\sin\theta \ , \tag{8}$$

which can be solved to give

$$\boxed{\Omega = \sqrt{\frac{g}{\frac{5}{3}R\tan\theta - r\sin\theta}} \ .} \tag{9}$$

SECTION II: ELECTRICITY & MAGNETISM

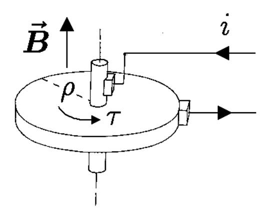
Electromagnetism 1: A Conducting Wheel in a Uniform Magnetic Field



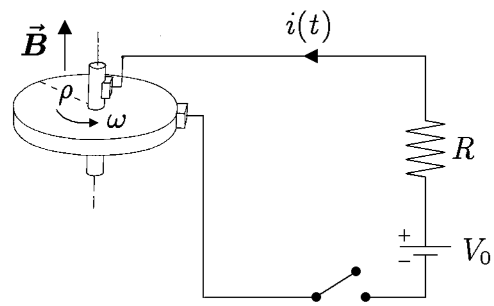
(a) (*3 pts*) A conducting wheel of radius ρ is pivoted so that it can rotate in the horizontal plane, in the presence of a uniform magnetic field of magnitude B in the vertical direction, as shown in the diagram. A wire connects to the shaft of the wheel through a frictionless sliding contact, and another wire connects to the outer edge of the wheel through another frictionless sliding contact. Neglect the diameter of the shaft. If the wheel is forced to rotate with angular frequency ω, counterclockwise as seen from above, an electrical potential difference will be generated between the two contacts. This is often called a homopolar generator. Calculate

$$\Delta V \equiv V_{\text{edge}} - V_{\text{center}} \tag{1}$$

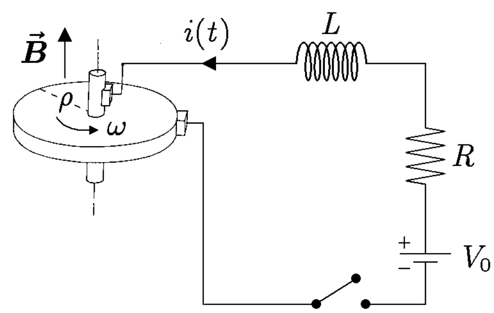
as a function of B, ρ, and ω.



(b) (*2 pts*) If a current i flows through the wheel, from the center to the edge, a torque τ is imparted to the wheel about its axis. Calculate τ, defined as positive if counterclockwise when viewed from above, as a function of B, ρ, and i. You may assume that the force that acts on the electrons is rapidly transferred to the solid body of the wheel.



(c) (*3 pts*) Suppose that the wires from the wheel are connected to a circuit, which also includes a switch, an ideal voltage source V_0, and a resistor R, as shown. Neglect all resistance in the wires, the wheel, and the contacts, neglect any self-inductance in the system, and neglect any friction. The moment of inertia of the wheel about its axis is I_0. Suppose that the switch is closed at $t = 0$, with the wheel initially at rest. Find the current $i_c(t)$ that flows through the circuit, and the angular velocity $\omega_c(t)$ of the wheel.



(d) (*2 pts*) Now suppose that an inductor of inductance L is added to the circuit, in series, as shown, with $L > R^2 I_0/(B^2\rho^4)$. The clock is reset, and the switch is again closed at $t = 0$, with the wheel at rest. For this case find the current $i_d(t)$ and the angular velocity $\omega_d(t)$ of the wheel.

SOLUTION:

(a) (*3 pts*) The unusual feature of this problem is the rotating wheel, which forces the electrons within it to have a nonzero velocity relative to the laboratory frame. For $\omega > 0$, this velocity is in the counter-clockwise tangential direction. In the presence of the magnetic field, there is a resulting force on the electrons of magnitude

$$F = |q|vB = |q|\omega r B \tag{2}$$

in the inward radial direction, where q is the charge of the electron. Since the electrical resistance of the wheel is negligible, this force will cause radial motion of the electrons, which will continue until an inward radial electric field E is established to cancel the radial force. Thus, inside the wheel,

$$\vec{\boldsymbol{E}} = -\omega r B \hat{\boldsymbol{r}} \ , \tag{3}$$

which creates a potential difference

$$\boxed{\Delta V = V_{\text{edge}} - V_{\text{center}} = \omega B \int_0^\rho r\, dr = \frac{1}{2}\omega B \rho^2 \ .} \tag{4}$$

(b) (*2 pts*) The current traveling through the wheel from inside to outside creates a torque on the wheel, about its axis, given by

$$\boxed{\tau = -iB \int_0^\rho r\, dr = -\frac{1}{2} i B \rho^2 \ .} \tag{5}$$

(c) (*3 pts*) Since $\tau = I_0\, d\omega/dt$,

$$\frac{d\omega}{dt} = -\frac{i(t) B \rho^2}{2 I_0} \ . \tag{6}$$

The circuit equation gives, for $t > 0$,

$$V_0 - i(t) R + \frac{1}{2}\omega(t) B \rho^2 = 0 \ . \tag{7}$$

Differentiating the above equation, we find

$$-\frac{di}{dt} R + \frac{1}{2} B \rho^2 \frac{d\omega}{dt} = 0 \ , \tag{8}$$

which with Eq. (6) implies that

$$-\frac{di}{dt} R - \frac{B^2 \rho^4}{4 I_0 R} i(t) = 0 \ . \tag{9}$$

The above equation implies that

$$i(t) = i(0) \exp\left\{-\frac{B^2\rho^4}{4I_0 R}t\right\} . \tag{10}$$

At $t = 0+$, immediately after the switch is thrown, $\omega = 0$, so the circuit equation (Eq. (7)) implies that

$$i(0) = V_0/R . \tag{11}$$

The above equation may appear to violate Eq. (9), but Eq. (9) holds only for $t > 0$, after the switch is closed. Thus, finally,

$$\boxed{i_c(t) = \frac{V_0}{R} \exp\left\{-\frac{B^2\rho^4}{4I_0 R}t\right\} .} \tag{12}$$

Then

$$\frac{d\omega}{dt} = -\frac{B\rho^2}{2I_0}\frac{V_0}{R} \exp\left\{-\frac{B^2\rho^4}{4I_0 R}t\right\} , \tag{13}$$

and given the initial value $\omega(0) = 0$, we have

$$\boxed{\omega_c(t) = -\frac{2V_0}{B\rho^2}\left[1 - \exp\left\{-\frac{B^2\rho^4}{4I_0 R}t\right\}\right] .} \tag{14}$$

(d) (*2 pts*) Eqs. (4) and (6) continue to hold, but the circuit equation in this case becomes

$$V_0 - i(t)R - L\frac{di}{dt} + \frac{1}{2}\omega(t)B\rho^2 = 0 . \tag{15}$$

Again we differentiate this equation, finding

$$-L\frac{d^2i}{dt^2} - R\frac{di}{dt} - \frac{B^2\rho^4}{4I_0}i(t) = 0 . \tag{16}$$

This is the equation for a damped harmonic oscillator, which we can solve by assuming a solution of the form $i(t) \propto e^{i\Omega t}$, where Ω is allowed to be complex. Then

$$L\Omega^2 - iR\Omega - \frac{B^2\rho^4}{4I_0} = 0 . \tag{17}$$

The roots of this quadratic equation are

$$\Omega = \frac{R}{2L}i \pm \Omega_R , \tag{18}$$

where

$$\boxed{\Omega_R = \frac{1}{2}\sqrt{\frac{B^2\rho^4}{I_0 L} - \frac{R^2}{L^2}} .} \tag{19}$$

For $L > R^2 I_0/(B^2\rho^4)$, Ω_R is real. The general solution to Eq. (16) consistent with the boundary condition $i(0) = 0$ is given by

$$i(t) = Ae^{-(R/2L)t} \sin \Omega_R t \ , \tag{20}$$

where A is a constant. The value of A can be determined by the initial condition for di/dt, which is found by using $i(0) = 0$ and $\omega(0) = 0$ in Eq. (15), which gives

$$\frac{di}{dt}(0) = \frac{V_0}{L} \ . \tag{21}$$

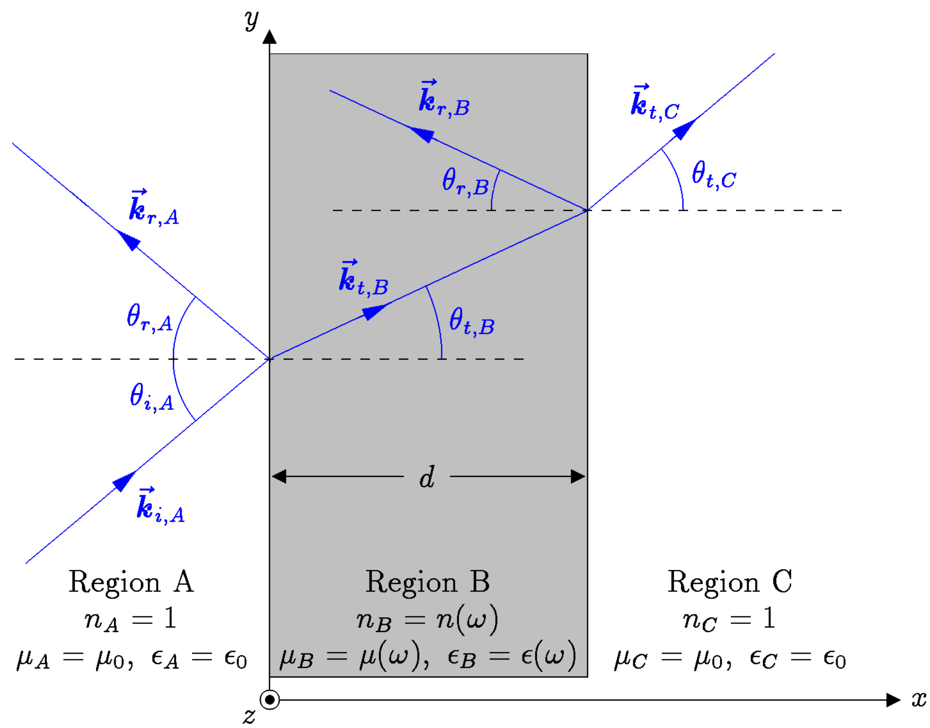
This gives

$$\boxed{i_c(t) = \frac{V_0}{L\Omega_R} e^{-(R/2L)t} \sin \Omega_R t \ .} \tag{22}$$

$\omega(t)$ can then be found by combining the above equation with Eq. (15), which gives

$$\boxed{\omega(t) = -\frac{2V_0}{B\rho^2}\left[1 - e^{-(R/2L)t} \cos \Omega_R t - \frac{R}{2L\Omega_R} e^{-(R/2L)t} \sin \Omega_R t\right] \ .} \tag{23}$$

Electromagnetism 2: Transmission of an EM Wave through a Dielectric Slab



Consider a dielectric slab of thickness d in empty space, with an index of refraction $n(\omega)$ which depends on the angular frequency ω of the radiation. The interior of the dielectric is called Region B, with Region A on its left and Region C on its right, as shown in the diagram. In Regions A and C the index of refraction is $n_A = n_C = 1$. The permittivity and permeability in Region B are given by $\epsilon(\omega)$ and $\mu(\omega)$, respectively, while in Regions A and C they have the vacuum values ϵ_0 and μ_0. Recall that the speed of light $c = 1/\sqrt{\mu_0\epsilon_0}$, and $n(\omega) = c\sqrt{\mu(\omega)\epsilon(\omega)}$.

In Region A there is an incident electromagnetic plane wave with angular frequency ω and propagation vector $\vec{k}_{i,A}$. The angle of incidence is $\theta_{i,A}$, and the wave is polarized so that the electric field $\vec{E}_{i,A}$ points out of the page, in the $\hat{z}$ direction. Explicitly,

$$\vec{E}_{i,A}(\vec{x},t) = \text{Re}\left\{E_{i,A}\hat{z}\ \exp\left[i\left(\vec{k}_{i,A}\cdot\vec{x} - \omega t\right)\right]\right\}\ , \tag{1}$$

where $E_{i,A}$ is a complex number. The electric field in all regions will point in the $\hat{z}$ direction.

(a) (*4 pts*) The electric field $\vec{E}_B(\vec{x},t)$ in Region B can be expressed approximately as the sum of a transmitted wave and a wave that is reflected from the back surface. As in Eq. (1), the transmitted wave can be written in terms of a complex number $E_{t,B}$ and a wave vector $\vec{k}_{t,B}$, where $E_{t,B}$ is proportional to $E_{i,A}$:

$$E_{t,B} = t_{BA}(\theta_{i,A})\, E_{i,A}\ . \tag{2}$$

Calculate the transmission function $t_{BA}(\theta_{i,A})$, ignoring the reflected wave in Region B. Find also the angle $\theta_{t,B}$ of the transmitted wave, measured from the normal, as shown.

Problem continued on next page.

(b) (*1 pt*) Similarly, we can write the electric field $\vec{\boldsymbol{E}}_{t,C}(\vec{\boldsymbol{x}}, t)$ of the transmitted wave in Region C in terms of a complex number $E_{t,C}$ and a wave vector $\vec{\boldsymbol{k}}_{t,C}$, where

$$E_{t,C} = t_{CB}(\theta_{t,B})\, E_{t,B} \ . \tag{3}$$

Calculate $t_{CB}(\theta_{t,B})$, and also the angle $\theta_{t,C}$ of the transmitted wave in Region C. Include only the first transit through the slab, ignoring contributions from reflections that pass through the slab more than once.

(c) (*1 pt*) In terms of $t_{BA}(\theta_{i,A})$ and $t_{CB}(\theta_{t,B})$, express the transmission coefficient $T(\theta_{i,A})$ (i.e., the fraction of *power* transmitted) from Region A to Region C.

(d) (*1 pt*) Now imagine replacing the dielectric slab by a dilute plasma occupying the same region. The plasma can be treated as a material with $\mu = \mu_0$ and dielectric constant

$$\epsilon(\omega) = \epsilon_0 \left[1 - \frac{\omega_p^2}{\omega^2} \right] \ , \tag{4}$$

where ω_p is the electron plasma frequency ($\omega_p^2 = 4\pi N e^2 / m_e$). For $\omega > \omega_p$, show that as the angle of incidence is increased it reaches a critical value $\theta_c(\omega_p/\omega)$ for which no transmission to Region C occurs.

(e) (*3 pts*) For $\theta_{i,A} > \theta_c$ (i.e., for an angle of incidence greater than critical), consider the fields inside Region B. Ignoring any reflections, calculate the ratio

$$R \equiv \lim_{\epsilon \to 0+} \frac{\text{Intensity of transmitted wave at } x = d - \epsilon}{\text{Intensity of transmitted wave at } x = \epsilon} \ , \tag{5}$$

where x is the horizontal coordinate with $x = 0$ at the A-B interface. In words, you should calculate the ratio of the transmitted wave intensity just inside the slab on the right to the intensity just inside the slab at the left.

SOLUTION:

(a) (*4 pts*) The essential facts about plane waves in a medium are that $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ can be written in terms of complex vectors $\vec{E}_0$ and $\vec{B}_0$ as

$$\vec{E}(\vec{x},t) = \text{Re}\left\{\vec{E}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}\right\} \tag{6}$$

$$\vec{B}(\vec{x},t) = \text{Re}\left\{\vec{B}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}\right\} , \tag{7}$$

where

$$\vec{B}_0 = \sqrt{\mu\epsilon}\,\hat{k}\times\vec{E}_0 = \frac{n}{c}\hat{k}\times\vec{E}_0 , \tag{8}$$

with

$$|\vec{k}| \equiv k = \omega\sqrt{\mu\epsilon} \tag{9}$$

and

$$\hat{k} \equiv \vec{k}/k . \tag{10}$$

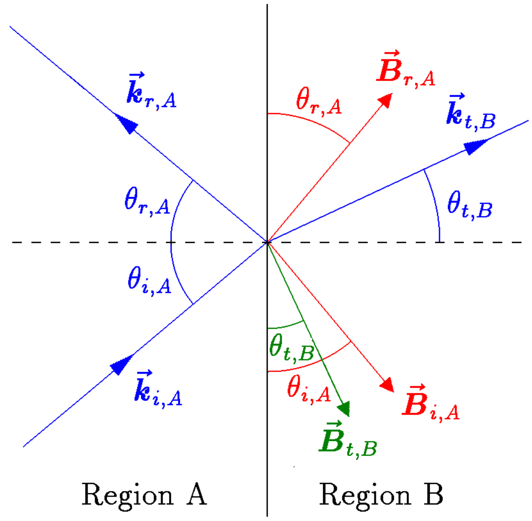
We will also use Snell's law, which says that if a plane wave travels from medium 1 to medium 2, the angles of the propagation direction in the two regions, measured from the normal to the interface, are related by

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 . \tag{11}$$

There is a reflected wave in medium 1, with an angle of reflection equal to the angle of incidence. We will need to know that at the boundary of two regions, the normal components of $\vec{D} = \epsilon\vec{E}$ and $\vec{B}$ are continuous, and the tangential components of $\vec{E}$ and $\vec{H} = \vec{B}/\mu$ are continuous. Finally, we will need to know that the time-averaged Poynting vector, which describes the flow of energy, is given by

$$\vec{S} = \frac{1}{2}\text{Re}(\vec{E}_0 \times \vec{H}_0^*) = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}|E_0|^2\hat{k} . \tag{12}$$

To apply these principles to the problem at hand, consider the transmission from Region A to Region B. The situation is shown in the following diagram:



The diagram shows the three plane waves that contribute to the situation: the incident wave in Region A (denoted by subscript i, A), the reflected wave in Region A (denoted by r, A), and the transmitted wave in Region B (denoted by t, B).

The electric field for all of the waves will point in the z direction, since there is nothing to produce an electric field in any other direction. The magnetic fields associated with each plane wave are shown on the diagram, with the direction fixed by Eq. (8). The complex vectors for the electromagnetic fields can then be written in terms of unit vectors as

$$\vec{\boldsymbol{E}}_{i,A} = E_{i,A}\,\hat{\boldsymbol{z}}\ , \quad \vec{\boldsymbol{B}}_{i,A} = B_{i,A}\,\hat{\boldsymbol{u}}_{i,A} = B_{i,A}\,\hat{\boldsymbol{k}}_{i,A} \times \hat{\boldsymbol{z}}\ , \tag{13}$$

$$\vec{\boldsymbol{E}}_{r,A} = E_{r,A}\,\hat{\boldsymbol{z}}\ , \quad \vec{\boldsymbol{B}}_{r,A} = B_{r,A}\,\hat{\boldsymbol{u}}_{r,A} = B_{r,A}\,\hat{\boldsymbol{k}}_{r,A} \times \hat{\boldsymbol{z}}\ , \tag{14}$$

$$\vec{\boldsymbol{E}}_{t,B} = E_{t,B}\,\hat{\boldsymbol{z}}\ , \quad \vec{\boldsymbol{B}}_{t,B} = B_{t,B}\,\hat{\boldsymbol{u}}_{t,B} = B_{t,B}\,\hat{\boldsymbol{k}}_{t,B} \times \hat{\boldsymbol{z}}\ . \tag{15}$$

We can now impose the required conditions. All angles can be related to $\theta_{i,A}$, since $\theta_{r,A} = \theta_{i,A}$ and Snell's law implies that

$$\sin\theta_{i,A} = n\sin\theta_{t,B} \quad \Longrightarrow \quad \boxed{\theta_{t,B} = \sin^{-1}\left(\frac{1}{n}\sin\theta_{i,A}\right)\ .} \tag{16}$$

Continuity of the tangential component of $\vec{\boldsymbol{E}}$ implies that

$$E_{i,A} + E_{r,A} = E_{t,B}\ . \tag{17}$$

Continuity of the normal component of $\vec{\boldsymbol{B}}$ is expressed by

$$(B_{i,A} + B_{r,A})\sin\theta_{i,A} = B_{t,B}\sin\theta_{t,B}\ , \tag{18}$$

where the trigonometric factors can be seen on the diagram. Using Eq. (8) to express the magnetic field in terms of the electric field, this relation can be rewritten as

$$(E_{i,A} + E_{r,A})\sin\theta_{i,A} = nE_{t,B}\sin\theta_{t,B}\ , \tag{19}$$

which, given Eq. (17), is equivalent to Snell's law. (The students are not expected to derive Snell's law, so it is fine if they say nothing about the normal component of $\vec{\boldsymbol{B}}$.) Continuity of the tangential component of $\vec{\boldsymbol{H}}$ is written as

$$(H_{i,A} - H_{r,A})\cos\theta_{i,A} = H_{t,B}\cos\theta_{t,B}\ , \tag{20}$$

which is equivalent to

$$(B_{i,A} - B_{r,A})\cos\theta_{i,A} = \frac{\mu_0}{\mu}B_{t,B}\cos\theta_{t,B}\ , \tag{21}$$

or

$$(E_{i,A} - E_{r,A})\cos\theta_{i,A} = \frac{\mu_0}{\mu}\,n\,E_{t,B}\cos\theta_{t,B}\ . \tag{22}$$

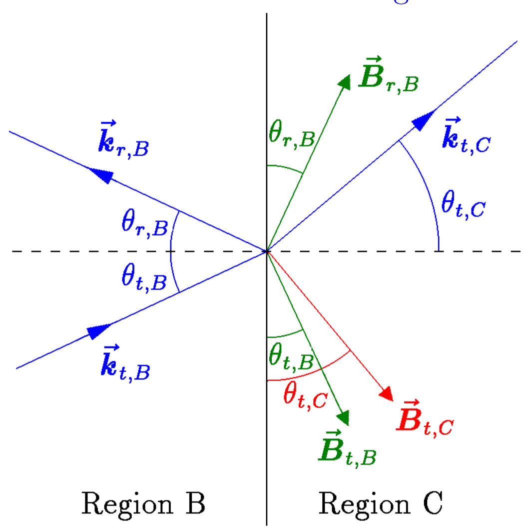
Manipulating Eq. (22),

$$
\begin{aligned}
E_{i,A} - E_{r,A} &= \frac{\mu_0}{\mu} n \frac{\cos\theta_{t,B}}{\cos\theta_{i,A}} E_{t,B} &(23)\\
&= \frac{\mu_0}{\mu} \frac{\sqrt{n^2 - n^2 \sin^2\theta_{t,B}}}{\cos\theta_{i,A}} E_{t,B} &(24)\\
&= \frac{\mu_0}{\mu} \frac{\sqrt{n^2 - \sin^2\theta_{i,A}}}{\cos\theta_{i,A}} E_{t,B} \ . &(25)
\end{aligned}
$$

By adding Eqs. (17) and (25), one can eliminate $E_{r,A}$ and then solve for $E_{t,B}$:

$$
E_{t,B} = t_{BA}(\theta_{i,A})\, E_{i,A} \ , \ \text{where} \ \boxed{t_{BA}(\theta_{i,A}) = \frac{2\mu \cos\theta_{i,A}}{\mu \cos\theta_{i,A} + \mu_0 \sqrt{n^2 - \sin^2\theta_{i,A}}} \ .} \qquad (26)
$$

(b) (*1 pt*) To find what happens at the interface between Regions B and C, one can use the following diagram, which is really just a relabeling of the diagram of the A-B interface. Here we need to introduce a reflected wave in Region B.



Here Snell's law becomes

$$
\sin\theta_{i,A} = n \sin\theta_{t,B} = \sin\theta_{t,C} \quad \Longrightarrow \quad \boxed{\theta_{t,C} = \theta_{i,A} \ .} \qquad (27)
$$

Continuity of the tangential component of $\vec{\boldsymbol{E}}$ gives

$$
E_{t_B} + E_{r,B} = E_{t,C} \ , \qquad (28)
$$

and continuity of the tangential component of $\vec{\boldsymbol{H}}$ gives

$$
(H_{t,B} - H_{r,B}) \cos\theta_{t,B} = H_{t,C} \cos\theta_{t,C} \ . \qquad (29)
$$

Again continuity of the normal component of $\vec{\boldsymbol{B}}$ will simply reproduce Snell's law. The algebra parallels the previous case, and the final result is

$$E_{t,C} = t_{CB}(\theta_{t,B})\, E_{t,B} \ , \ \text{where} \ \boxed{t_{CB}(\theta_{t,B}) = \frac{2\mu_0 \cos\theta_{t,B}}{\mu_0 \cos\theta_{t,B} + \mu\sqrt{1/n^2 - \sin^2\theta_{t,B}}} \ ,} \tag{30}$$

which can alternatively be written as

$$\boxed{t_{CB}(\theta_{t,B}) = \frac{2\mu_0\sqrt{n^2 - \sin^2\theta_{i,A}}}{\mu_0\sqrt{n^2 - \sin^2\theta_{i,A}} + \mu\cos\theta_{i,A}} \ .} \tag{31}$$

(c) (*1 pt*) The power transmitted (Poynting flux) is proportional to $|E|^2$, so

$$\boxed{T(\theta_{i,A}) = |t_{BA}(\theta_{i,A}) t_{CB}(\theta_{t,B})|^2 \ .} \tag{32}$$

Since Regions A and C contain the same medium, we do not need to include any dependence on ϵ or μ.

The problem does not ask for a detailed answer in terms of the original variables, but if students supply such an answer, it should be

$$T(\theta_{i,A}) = \frac{16\mu_0^2\mu^2\left(n^2 - \sin^2\theta_{i,A}\right)\cos^2\theta_{i,A}}{\left[\mu_0\sqrt{n^2 - \sin^2\theta_{i,A}} + \mu\cos\theta_{i,A}\right]^4} \ . \tag{33}$$

(d) (*1 pt*) Since $n(\omega) = c\sqrt{\mu(\omega)\epsilon(\omega)}$, and $\mu = \mu_0$ and $\epsilon(\omega) = \epsilon_0\sqrt{1 - (\omega_p/\omega)^2}$, we have

$$n(\omega) = \sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2} \ . \tag{34}$$

Thus $n(\omega)$ is less than one, which allows $t_{CB}(\theta_{t,B})$ to vanish, as can be seen by looking at the right-hand-side of Eq. (31), which vanishes when $\sin\theta_{i,A} = n$. Thus, the critical angle is

$$\boxed{\theta_c = \sin^{-1}\left(\sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}\right) \ .} \tag{35}$$

(The students were not asked anything about the fields, but in case the students mention them the grader should be aware that $t_{BA}(\theta_{i,A})$ does not vanish, so the fields do not vanish in Region B. But $\theta_{t,B} = 90°$, so k_x vanishes and the fields do not change with x. Since $\vec{\boldsymbol{k}}_{t,B}$ points in the y direction, the $\vec{\boldsymbol{B}}$ field points in the x direction, implying that the Poynting vector has no component in the x direction.)

(e) (*3 pts*) As usual we try a solution of the form

$$\begin{aligned} \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t) &= \mathrm{Re}\left\{ \vec{\boldsymbol{E}}_{t,B} e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega t)} \right\} , \\ \vec{\boldsymbol{B}}(\vec{\boldsymbol{x}},t) &= \mathrm{Re}\left\{ \vec{\boldsymbol{B}}_{t,B} e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega t)} \right\} , \end{aligned} \tag{36}$$

where $\vec{\boldsymbol{k}} \equiv \vec{\boldsymbol{k}}_{t,B}$. In this case we will find that $\vec{\boldsymbol{k}}$ will need to be complex.

For the continuity relations to hold for all y, we must have

$$k_y = k_{y,i,A} = \frac{\omega}{c} \sin\theta_{i,A} \ . \tag{37}$$

To solve the wave equation we must also have $|\vec{\boldsymbol{k}}| = n\omega/c$, so

$$k_x = \frac{\omega}{c}\sqrt{n^2 - \sin^2\theta_{i,A}} \ . \tag{38}$$

But this square root has an ambiguous sign. The choice will correspond to either a rising or a falling solution as a function of x, and both are valid solutions to Maxwell's equations. However, given the boundary condition that the wave is coming in from the left, only the solution that falls with x is possible. Thus,

$$\vec{\boldsymbol{k}} = \frac{\omega}{c}\left[i\sqrt{\sin^2\theta_{i,A} - n^2}\,\hat{\boldsymbol{x}} + \sin\theta_{i,A}\,\hat{\boldsymbol{y}} \right] \ . \tag{39}$$

Thus $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ are proportional to $\exp\left(-\frac{\omega}{c}\sqrt{\sin^2\theta_{i,A} - n^2}\,x\right)$, whle the intensity is proportional to $\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}$. Thus,

$$\boxed{R = e^{-2\frac{\omega d}{c}\sqrt{\sin^2\theta_{i,A} - n^2}}} \ . \tag{40}$$

Statistical Mechanics 1: A Strongly Interacting Fermi Gas

A spin-1/2 Fermi gas with (s-wave) attractive interactions between spin-up and spin-down fermions forms a superfluid of fermion pairs at low temperatures. When these interactions are resonant for s-wave scattering—i.e. the scattering cross section saturates the unitarity bound—only two intrinsic energy scales are relevant for describing the system: the energy scale associated with temperature $k_B T$, and the Fermi energy $E_F = \hbar^2 k_F^2/2m$. Here, $k_F = (3\pi^2 n)^{1/3}$ is the Fermi wavevector, $n = N/V$ is the total density, N is the total number of fermions, V is the volume of the gas, and m is the fermion mass.

At zero temperature, the energy scale $k_B T$ is irrelevant, so the total energy of the gas *must* be a universal number ξ times the ground state energy of a non-interacting Fermi gas,

$$E_{\text{tot}}|_{T=0} = \frac{3}{5}\xi N E_F. \tag{1}$$

Since the pressure $P = \partial E/\partial V$ at $T = 0$, the relationship $P = \frac{2}{3}E/V$ holds just as for a non-interacting Fermi gas. The pressure of the strongly interacting Fermi gas is thus the same universal number ξ times the pressure for a non-interacting Fermi gas,

$$P = \frac{2}{5}\xi n E_F. \tag{2}$$

In this problem, you will show that not only the zero-temperature but also the nonzero-temperature thermodynamic properties of the system are uniquely specified by ξ as long as the temperature is low enough not to break any fermion pairs.

(a) (*1 pt*) At low enough temperatures, the only excitations of the gas are phonons, i.e. sound waves. From the known equation of state between pressure and density at zero temperature, calculate the speed of sound c for phonons in the gas. Express your result in terms of ξ and the Fermi velocity $v_F = \hbar k_F/m$. For the remainder of this problem, you may assume that this value of c holds at all temperatures of interest. [*Hint:* recall that

$$c^2 = \frac{\partial P}{\partial \rho}\ , \tag{3}$$

where ρ is the mass per unit volume.]

Problem continued on next page.

(b) (*5 pts*) Find the contribution from phonons to the free energy of the gas

$$F_{\text{ph}}(N,V,T) \equiv -k_B T \ln Z_{\text{ph}}, \tag{4}$$

where Z_{ph} is the partition function for phonons

$$Z_{\text{ph}} \equiv \sum_{\text{all ph states}} e^{-E_{\text{ph}}/k_B T}, \tag{5}$$

and the sum is over all states involving any number of phonons (but no other excitations). Write your result in the form

$$F_{\text{ph}}(N,V,T) = a\, T^{\alpha} N^{\beta} V^{\gamma} \xi^{\delta} \tag{6}$$

where you need to find the (dimensionful) constant a and exponents α, β, γ, and δ.

Even if you do not succeed in finding a, α, β, γ, and δ, you can do all subsequent parts by expressing your answers to each of them in terms of these constants.

[*Hint:* The energy of a single-phonon state is

$$E_{\text{1-ph}} = \hbar c |\vec{\boldsymbol{k}}_{\text{1-ph}}|, \tag{7}$$

where $\vec{\boldsymbol{k}}_{\text{1-ph}}$ is the phonon wave vector, and c is the speed of sound calculated in part (a). Neglect any phonon interactions, so the total phonon energy is the sum of the single phonon energies:

$$E_{\text{tot-ph}} = \sum_i \hbar c |\vec{\boldsymbol{k}}_i| \ . \tag{8}$$

You can assume that the gas occupies a cube of volume V with periodic boundary conditions, and you can assume the volume is large enough to replace various sums by integrals. In addition, you can neglect the possibility of phonons decaying into broken fermion pairs, allowing momentum integrals to be extended to infinitely large momenta. You may find the following integral to be useful

$$\int_0^{\infty} dx\, x^2 \ln(1 - e^{-x}) = -\frac{\pi^4}{45}. \qquad] \tag{9}$$

(c) (*2 pts*) Calculate the phonon contribution $S_{\text{ph}}(N,V,T)$ to the entropy and the phonon contribution $E_{\text{ph}}(N,V,T)$ to the total energy of the Fermi gas.

(d) (*1 pt*) Suppose that the low-temperature gas, initially at temperature T_0 and volume V_0, is now adiabatically compressed. If the volume shrinks to $V_f = \epsilon V_0$ with $0 < \epsilon < 1$, what is the final temperature T_f?

(e) (*1 pt*) Calculate the specific heat at constant volume C_V of the gas.

(f) (*0 pts*) *You do not need to do this part!* But if you wish, you can check your answers to parts (b), (c), and (e) by verifying that $F_{\text{ph}}/(NE_F)$, $E_{\text{ph}}/(NE_F)$, $S_{\text{ph}}/(Nk_B)$ and $C_V/(Nk_B)$ can each be written as functions of $(k_B T/E_F)$.

SOLUTION:

(a)

$$c^2 = \frac{1}{m}\frac{\partial P}{\partial n} = \frac{5}{3}\frac{P}{mn} = \frac{2}{3}\xi\frac{E_F}{m} = \frac{2}{3}\xi\frac{\hbar^2 k_F^2}{2m^2} \tag{10}$$

$$c = \boxed{\sqrt{\frac{\xi}{3}}\, v_F\ .} \tag{11}$$

(b) The possible phonon states have any number of phonons n_i in each of the momentum states, labeled $\vec{\boldsymbol{k}}_i$, so

$$Z_{\text{ph}} = \sum_{\{n_i\}} e^{-\hbar c \sum n_i |\vec{\boldsymbol{k}}_i|/k_B T} \tag{12}$$

$$= \sum_{\{n_i\}} \prod_i e^{-\hbar c n_i |\vec{\boldsymbol{k}}_i|/k_B T} \tag{13}$$

$$= \prod_i \sum_n e^{-\hbar c n |\vec{\boldsymbol{k}}_i|/k_B T} \tag{14}$$

$$= \prod_i \frac{1}{1 - e^{-\hbar c |\vec{\boldsymbol{k}}_i|/k_B T}} \tag{15}$$

$$= \prod_{k_x}\prod_{k_y}\prod_{k_z} \frac{1}{1 - e^{-\hbar c |\vec{\boldsymbol{k}}|/k_B T}}\ . \tag{16}$$

Then

$$F = -k_B T \ln Z_{\text{ph}} = k_B T \sum_{\vec{\boldsymbol{k}}} \ln\left(1 - e^{-\hbar c|\vec{\boldsymbol{k}}|/k_B T}\right) \tag{17}$$

$$= k_B T\, V \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \ln\left(1 - e^{-\hbar c|\vec{\boldsymbol{k}}|/k_B T}\right) \tag{18}$$

$$= \frac{V}{2\pi^2} k_B T \int_0^\infty \mathrm{d}k\; k^2 \ln\left(1 - e^{-\hbar c k/k_B T}\right) \tag{19}$$

$$= \frac{V}{2\pi^2}\frac{(k_B T)^4}{(\hbar c)^3} \int_0^\infty \mathrm{d}x\; x^2 \ln\left(1 - e^{-x}\right) \tag{20}$$

$$= -\frac{\pi^2}{90}\frac{V (k_B T)^4}{(\hbar c)^3}\ . \tag{21}$$

We have $c = \sqrt{\frac{\xi}{3}}\frac{\hbar k_F}{m}$, and $k_F^3 = 3\pi^2 N/V$, so the factor $1/c^3$ in the denominator becomes

$$\frac{1}{c^3} = \frac{3^{3/2}}{\xi^{3/2}}\frac{m^3}{\hbar^3}\frac{V}{3\pi^2 N}\ , \tag{22}$$

and thus

$$
\begin{aligned}
F_{\rm ph} &= -\frac{\pi^2}{90}\frac{V(k_BT)^4}{(\hbar c)^3} && (23)\\
&= -\frac{\pi^2}{90}\frac{3^{3/2}}{3\pi^2\xi^{3/2}}\frac{m^3}{\hbar^6}\frac{V^2(k_BT)^4}{N} && (24)\\
&= \boxed{-\frac{\sqrt{3}}{90\xi^{3/2}}\frac{m^3}{\hbar^6}\frac{V^2(k_BT)^4}{N}\ .} && (25)
\end{aligned}
$$

So $\boxed{a = -\frac{\sqrt{3}}{90}\frac{m^3k_B^4}{\hbar^6},\ \alpha = 4,\ \beta = -1,\ \gamma = 2 \text{ and } \delta = -3/2.}$

(c)

$$
\begin{aligned}
S_{\rm ph} &= -\left.\frac{\partial F}{\partial T}\right|_{N,V} = -4\frac{F_{\rm ph}}{T} && (26)\\
&= \boxed{k_B\frac{2\sqrt{3}}{45\xi^{3/2}}\frac{m^3}{\hbar^6}\frac{V^2(k_BT)^3}{N}\ .} && (27)
\end{aligned}
$$

Note that this can be written via the Fermi energy $E_F = \frac{\hbar^2k_F^2}{2m} = \frac{\hbar^2(3\pi^2N)^{2/3}}{2mV^{2/3}}$ as

$$
\begin{aligned}
S_{\rm ph} &= Nk_B\frac{2\sqrt{3}(3\pi^2)^2}{2^3 45\xi^{3/2}}\frac{2^3m^3}{\hbar^6}\frac{V^2(k_BT)^3}{(3\pi^2)^2N^2} && (28)\\
&= Nk_B\frac{\sqrt{3}\pi^4}{20\xi^{3/2}}\left(\frac{k_BT}{E_F}\right)^3 . && (29)
\end{aligned}
$$

(This form is not required for full credit.)

In terms of a, α, β, γ and δ:

$$
\begin{aligned}
S_{\rm ph} &= -\left.\frac{\partial F}{\partial T}\right|_{N,V} = -\alpha\frac{F_{\rm ph}}{T} && (30)\\
&= \boxed{-a\,\alpha\,T^{\alpha-1}N^\beta V^\gamma \xi^\delta\ .} && (31)
\end{aligned}
$$

We have

$$
E_{\rm ph} = F_{\rm ph} + TS_{\rm ph} = F_{\rm ph} - 4F_{\rm ph} = -3F_{\rm ph} = \boxed{NE_F\frac{3\sqrt{3}\pi^4}{80\xi^{3/2}}\left(\frac{k_BT}{E_F}\right)^4 ,} \qquad (32)
$$

or equivalently

$$E_{\rm ph} = \boxed{\frac{\sqrt{3}}{30\xi^{3/2}}\frac{m^3}{\hbar^6}\frac{V^2(k_BT)^4}{N}\ .} \tag{33}$$

In terms of a, α, β, γ and δ:

$$E_{\rm ph} = F_{\rm ph} + TS_{\rm ph} = F_{\rm ph} - \alpha F_{\rm ph} = (1-\alpha)F_{\rm ph} = \boxed{(1-\alpha)\, a\, T^\alpha N^\beta V^\gamma \xi^\delta\ .} \tag{34}$$

(d) As the entropy $S_{\rm ph} \propto V^2T^3$, we have for adiabatic compression $T \propto V^{-2/3}$ and thus

$$\boxed{T_f = T_0\frac{1}{\epsilon^{2/3}}\ .} \tag{35}$$

Another way to see this is to note that $S_{\rm ph}/Nk_B$ is a dimensionless number, and $S_{\rm ph}/Nk_B$ must thus go like some function of k_BT/E_F, as this is the only way to cancel out the two available energy scales. So $S_{\rm ph}$ = const. at fixed particle number implies $T \propto T_F \propto V^{-2/3}$.

In terms of a, α, β, γ and δ: As the entropy $S_{\rm ph} \propto V^\gamma T^{\alpha-1}$, we have for adiabatic compression $T \propto V^{-\frac{\gamma}{\alpha-1}}$ and thus

$$\boxed{T_f = T_0\epsilon^{-\frac{\gamma}{\alpha-1}}\ .} \tag{36}$$

(e)

$$C_V = \left.\frac{\partial E}{\partial T}\right|_{N,V} = 4\frac{E_{\rm ph}}{T} = -12\frac{F_{\rm ph}}{T} = \boxed{\frac{2\sqrt{3}}{15\xi^{3/2}}\frac{m^3}{\hbar^6}\frac{V^2(k_BT)^4}{N}\ ,} \tag{37}$$

or equivalently

$$C_V = \boxed{Nk_B\frac{3\sqrt{3}\pi^4}{20\xi^{3/2}}\left(\frac{k_BT}{E_F}\right)^3\ .} \tag{38}$$

One can thus write $C_V = Nk_B f_C(k_BT/E_F)$, where

$$\boxed{f_C(x) = \frac{3\sqrt{3}\pi^4}{20\xi^{3/2}}x^3\ .} \tag{39}$$

(This form is not required for full credit.)

In terms of a, α, β, γ and δ:

$$C_V = \left.\frac{\partial E}{\partial T}\right|_{N,V} = \alpha\frac{E_{\rm ph}}{T} = \boxed{(1-\alpha)\,\alpha\, a\, T^{\alpha-1}N^\beta V^\gamma \xi^\delta\ .} \tag{40}$$

Statistical Mechanics 2: Interaction between Dipole Moments

Two infinitely heavy classical particles are separated by a distance $r = |\vec{\boldsymbol{r}}|$. Each particle has a dipole moment $\vec{\boldsymbol{M}} = (M_x, M_y, M_z)$ with $M_x^2 + M_y^2 + M_z^2 = M^2$ being fixed and equal for both particles, and both dipoles are free to rotate. The interaction energy between the dipole moments is

$$V = \frac{3(\vec{\boldsymbol{M}}_1 \cdot \hat{\boldsymbol{r}})(\vec{\boldsymbol{M}}_2 \cdot \hat{\boldsymbol{r}}) - (\vec{\boldsymbol{M}}_1 \cdot \vec{\boldsymbol{M}}_2)}{r^3} , \tag{1}$$

where $\hat{\boldsymbol{r}}$ is the unit vector in the $\vec{\boldsymbol{r}}$ direction. The system is in thermal equilibrium with the environment at high temperature T, such that

$$k_B T \gg \frac{M^2}{r^3}. \tag{2}$$

(a) (*1 pt*) Assume that the dipoles are separated along the $\hat{\boldsymbol{z}}$ axis. Write the interaction energy as a function of the angular variables (θ_1,ϕ_1) and (θ_2,ϕ_2), where

$$\vec{\boldsymbol{M}}_i = M(\sin\theta_i \cos\phi_i, \sin\theta_i \sin\phi_i, \cos\theta_i) \tag{3}$$

specify the dipole directions in spherical coordinates.

(b) (*4 pts*) Find an expression for the partition function $Z(r)$ at *fixed* r. Evaluate the partition function in the high temperature limit given above, up to second order in $\gamma(r) = \frac{M^2}{k_B T r^3}$. Some potentially useful integrals are:

$$\int_0^\pi \mathrm{d}\theta \ \sin^2\theta = \frac{\pi}{2}, \qquad \int_0^\pi \mathrm{d}\theta \ \sin^3\theta = \frac{4}{3}, \qquad \int_0^\pi \mathrm{d}\theta \ \sin^4\theta = \frac{3\pi}{8}. \tag{4}$$

(c) (*2 pts*) Compute the free energy $F(r)$ and internal energy $E(r)$ of the dipole-dipole system up to the lowest non-trivial term in $\gamma(r)$.

(d) (*1 pt*) What is the average force $\vec{\boldsymbol{f}}(r)$ between the particles at a distance r?

(e) (*1 pt*) Now assume that the particles are connected by a spring with elastic energy

$$U(r) = \frac{1}{2}A(R-r)^2, \tag{5}$$

where A and R are constants. Calculate the equilibrium separation between the particles, assuming that

$$\frac{1}{2}AR^2 \gg \frac{M^2}{R^3}, k_B T; \tag{6}$$

i.e. $U \gg V, k_B T$.

(f) (*1 pt*) Using the above result, calculate the coefficient of linear expansion

$$\alpha \equiv \frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}T}. \tag{7}$$

SOLUTION:

(a) Taking the z-axis along the line connecting the particles and calling $\theta_{1,2}$ the angles between $\vec{\boldsymbol{M}}_{1,2}$ and the z-axis we have

$$(\vec{\boldsymbol{M}}_{1,2} \cdot \vec{r}) = Mr \cos\theta_{1,2} \ , \tag{8}$$

$$(\vec{\boldsymbol{M}}_2 \cdot \vec{\boldsymbol{M}}_1) = M^2 \left(\cos\theta_2 \cos\theta_1 + \sin\theta_2 \sin\theta_1 \cos(\phi_2 - \phi_1)\right) \ , \tag{9}$$

where $\phi_{1,2}$ is the azimuthal angle of $\vec{\boldsymbol{M}}_{1,2}$. As a result,

$$\boxed{V = \frac{M^2}{r^3} \left(2\cos\theta_2 \cos\theta_1 - \sin\theta_2 \sin\theta_1 \cos(\phi_2 - \phi_1)\right) \ .} \tag{10}$$

(b) The classical partition function is

$$Z = \int \mathrm{d}\Omega_1 \mathrm{d}\Omega_2 \exp(-V/k_B T) \tag{11}$$

$$= \int \sin\theta_1 \mathrm{d}\theta_1 \mathrm{d}\phi_1 \int \sin\theta_2 \mathrm{d}\theta_2 \mathrm{d}\phi_2 \times \tag{12}$$

$$\exp\left\{ \left(-\frac{M^2}{k_B T r^3} \right) \left(2\cos\theta_2 \cos\theta_1 - \sin\theta_2 \sin\theta_1 \cos(\phi_2 - \phi_1)\right) \right\} \ . \tag{13}$$

At high temperatures $k_B T \gg \frac{M^2}{R^3} \approx \frac{M^2}{r^3}$ we can expand the exponent. The term linear in $1/k_B T$ vanishes after the angular integration in Eq. (13). Proceeding to second order in

$$\boxed{\gamma(r) = \frac{M^2}{k_B T r^3} \ ,} \tag{14}$$

we obtain

$$Z = \int \sin\theta_1 \mathrm{d}\theta_1 \mathrm{d}\phi_1 \int \sin\theta_2 \mathrm{d}\theta_2 \mathrm{d}\phi_2 \times \tag{15}$$

$$\left\{ 1 + \frac{1}{2} \left(\frac{M^2}{k_B T r^3} \right)^2 \left(2\cos\theta_2 \cos\theta_1 - \sin\theta_2 \sin\theta_1 \cos(\phi_2 - \phi_1)\right)^2 \right\} \ . \tag{16}$$

Evaluation of the integral yields

$$Z = 16\pi^2 + \frac{1}{2}\gamma(r)^2 \left\{ \left(\frac{8\pi}{3} \right)^2 + \frac{1}{2} \left(\frac{8\pi}{3} \right)^2 \right\} \tag{17}$$

$$= 16\pi^2 + 48 \left(\frac{\pi}{3} \gamma(r) \right)^2 \tag{18}$$

$$= \boxed{16\pi^2 \left(1 + \frac{1}{3}\gamma(r)^2 \right) \ .} \tag{19}$$

(c) For the free energy we have

$$F = -k_B T \ln Z = -k_B T \ln \left[16\pi^2 \left(1 + \frac{1}{3}\gamma(r)^2\right)\right] \quad (20)$$

$$\simeq \boxed{-k_B T \ln(16\pi^2) - \frac{1}{3} k_B T \gamma(r)^2 \ .} \quad (21)$$

For the internal energy we have

$$E = -\frac{\mathrm{d} \ln Z}{\mathrm{d}\left(\frac{1}{k_B T}\right)} \quad (22)$$

$$\simeq -\frac{2}{3}\gamma(r)\frac{\mathrm{d}\gamma(r)}{\mathrm{d}\left(\frac{1}{k_B T}\right)} \quad (23)$$

$$= \boxed{-\frac{2}{3}\left(\frac{M^2}{r^3}\right)^2 \frac{1}{k_B T} \ .} \quad (24)$$

(d) In analogy to the pressure of a gas, the force is the negative change of the *free* energy with respect to r at constant temperature. Note that $dE = T\,dS - f\ dr$, so the force would be $f = -\left.\frac{\partial E}{\partial r}\right|_S$, the derivative at constant entropy, so it would be a mistake to differentiate E at constant T.

$$\vec{f} = -\nabla F = \quad (25)$$

$$\simeq k_B T \frac{1}{3} \nabla \left(\gamma(r)^2\right) \quad (26)$$

$$= k_B T \frac{2}{3}\gamma(r)\hat{r}\frac{\mathrm{d}\gamma(r)}{\mathrm{d}r} \quad (27)$$

$$= \boxed{-2\frac{M^4}{r^7}\frac{1}{k_B T}\hat{r} \ .} \quad (28)$$

(e) The total free energy can be written as

$$F_{\text{tot}} = F + U(r) \quad (29)$$

$$= \frac{1}{2}A(r-R)^2 - \frac{1}{3}\left(\frac{M^2}{r^3}\right)^2 \frac{1}{k_B T} - k_B T \ln\left(16\pi^2\right) \ . \quad (30)$$

F_{tot} has a minimum at $\boxed{r = R - \delta r,}$ with

$$\boxed{\delta r = \frac{2M^4}{AR^7 k_B T} \ .} \quad (31)$$

(f) For the coefficient of linear expansion we finally obtain

$$\alpha = -\frac{1}{R}\frac{\mathrm{d}\delta r}{\mathrm{d}T} = \frac{2M^4}{AR^8 k_B T^2} \tag{32}$$

$$= \boxed{k_B \left(\frac{M^2}{k_B T R^3}\right)^2 \frac{2}{AR^2}\ .} \tag{33}$$

SECTION IV: QUANTUM MECHANICS

Quantum Mechanics 1: A Heisenberg Ferromagnet

In a ferromagnetic material, the electron spins are aligned, suggesting that an interaction of the form

$$\delta H = \kappa \vec{\boldsymbol{S}}_1 \cdot \vec{\boldsymbol{S}}_2 \tag{1}$$

is present between each pair of electrons, with $\kappa < 0$, where $\vec{\boldsymbol{S}}_1$ and $\vec{\boldsymbol{S}}_2$ are the operators corresponding to the spins of the two electrons. While Eq. (1) does not appear explicitly in the Hamiltonian for a ferromagnet, Heisenberg realized that Eq. (1) could appear as an *effective* interaction, arising from Coulomb repulsion and the fermionic properties of electrons.

In this problem, you will derive Heisenberg ferromagnetism for two spin-1/2 electrons in a common potential $V(\vec{\boldsymbol{r}})$, with Hamiltonian

$$H = \frac{|\vec{\boldsymbol{p}}_1|^2}{2m} + \frac{|\vec{\boldsymbol{p}}_2|^2}{2m} + V(\vec{\boldsymbol{r}}_1) + V(\vec{\boldsymbol{r}}_2) + \frac{e^2}{|\vec{\boldsymbol{r}}_1 - \vec{\boldsymbol{r}}_2|}. \tag{2}$$

Note that δH above is not included in this Hamiltonian, so there is no explicit spin dependence. The single-particle Schrödinger equation with potential $V(\vec{\boldsymbol{r}})$ has eigenstates with energies E_i and wave functions $\psi_i(\vec{\boldsymbol{r}})$.

(a) (*1 pt*) The total wave function $\Psi(\vec{\boldsymbol{r}}_1, \{s_1\}; \vec{\boldsymbol{r}}_2, \{s_2\})$ depends on the set of spin variables $\{s_1\}$ and $\{s_2\}$. (You may use whichever notation for spin that you prefer, including bra/ket notation.) Show that the eigenstates of the Hamiltonian can be written in a separable form as $\psi(\vec{\boldsymbol{r}}_1, \vec{\boldsymbol{r}}_2)\chi(\{s_1\}, \{s_2\})$. Construct eigenstates of total spin, and describe the symmetry properties of $\psi(\vec{\boldsymbol{r}}_1, \vec{\boldsymbol{r}}_2)$ under particle exchange for each of the spin states.

(b) (*1 pt*) In the absence of Coulomb repulsion, consider the two-particle configurations where one electron is in state $\psi_a(\vec{\boldsymbol{r}})$ and the other electron is in state $\psi_b(\vec{\boldsymbol{r}})$. What are the corresponding two-particle wave functions, including spin?

(c) (*3 pts*) The degeneracy of the states in part (b) is broken by Coulomb repulsion, yielding two distinct energy levels. Identify the two-particle wave functions associated with these two levels, and find an expression for the energy splitting δE to first order in the Coulomb interaction in terms of ψ_a and ψ_b.

(d) (*3 pts*) Show that the energy splitting in part (c) can be mimicked (at first order) by turning off the Coulomb interaction in Eq. (2) and replacing it with Eq. (1). Find an expression for κ in terms of δE.

(e) (*2 pts*) Determine the sign of κ. [*Hint:* the judicious use of Fourier transforms may be helpful. Recall that

$$\int d^3r\, e^{i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{r}}} = (2\pi)^3 \delta^{(3)}(\vec{\boldsymbol{q}}) \; ; \qquad \int d^3r\, e^{i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{r}}} \frac{1}{|\vec{\boldsymbol{r}}|} = \frac{4\pi}{|\vec{\boldsymbol{q}}|^2} \; .] \tag{3}$$

SOLUTION:

(a) Because the Hamiltonian does not depend explicitly on spin, H commutes with $\vec{\boldsymbol{S}}_1$ and $\vec{\boldsymbol{S}}_2$, allowing us to factorize the energy eigenfunctions as $\psi(\vec{\boldsymbol{r}}_1, \vec{\boldsymbol{r}}_2)\chi(\{s_1\}, \{s_2\})$. Using the notation $|s, m\rangle$, the eigenstates of total spin $s = 1$ are

$$|1, -1\rangle = |\downarrow\downarrow\rangle\,, \qquad |1, 0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right), \qquad |1, +1\rangle = |\uparrow\uparrow\rangle\,, \tag{4}$$

and the eigenstate of total spin 0 is

$$|0, 0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right). \tag{5}$$

Because the overall fermionic wave function has to be antisymmetric under the interchange of particles 1 and 2, the wave functions must take the form

$$\psi_A(\vec{\boldsymbol{r}}_1, \vec{\boldsymbol{r}}_2)\,|1, m\rangle\,, \qquad \psi_S(\vec{\boldsymbol{r}}_1, \vec{\boldsymbol{r}}_2)\,|0, 0\rangle\,, \tag{6}$$

where ψ_A is antisymmetric under interchange, and ψ_S is symmetric under interchange.

(b) Using the notation of Eq. (6), the antisymmetric ($s = 1$) configuration is

$$\psi_A(\vec{\boldsymbol{r}}_1, \vec{\boldsymbol{r}}_2) = \frac{1}{\sqrt{2}}\left(\psi_a(\vec{\boldsymbol{r}}_1)\psi_b(\vec{\boldsymbol{r}}_2) - \psi_b(\vec{\boldsymbol{r}}_1)\psi_a(\vec{\boldsymbol{r}}_2)\right) \tag{7}$$

and the symmetric ($s = 0$) configuration is

$$\psi_S(\vec{\boldsymbol{r}}_1, \vec{\boldsymbol{r}}_2) = \frac{1}{\sqrt{2}}\left(\psi_a(\vec{\boldsymbol{r}}_1)\psi_b(\vec{\boldsymbol{r}}_2) + \psi_b(\vec{\boldsymbol{r}}_1)\psi_a(\vec{\boldsymbol{r}}_2)\right). \tag{8}$$

(c) The states $\psi_A(\vec{\boldsymbol{r}}_1, \vec{\boldsymbol{r}}_2)\,|1, m\rangle$ and $\psi_S(\vec{\boldsymbol{r}}_1, \vec{\boldsymbol{r}}_2)\,|0, 0\rangle$ are degenerate, so in general, one would have to diagonalize the perturbation in order to use perturbation theory. However, these states do diagonalize the Coulomb perturbation, since ψ_A and ψ_S have different symmetry properties under interchange while the Coulomb perturbation is symmetric, such that

$$\langle\psi_A|\,\frac{e^2}{|\vec{\boldsymbol{r}}_1 - \vec{\boldsymbol{r}}_2|}\,|\psi_S\rangle = 0. \tag{9}$$

Thus, $\psi_A(\vec{\boldsymbol{r}}_1, \vec{\boldsymbol{r}}_2)\,|1, m\rangle$ and $\psi_S(\vec{\boldsymbol{r}}_1, \vec{\boldsymbol{r}}_2)\,|0, 0\rangle$ correspond to the two energy levels. Their energy difference is

$$\delta E \equiv E_{s=1} - E_{s=0} = \langle\psi_A|\,\frac{e^2}{|\vec{\boldsymbol{r}}_1 - \vec{\boldsymbol{r}}_2|}\,|\psi_A\rangle - \langle\psi_S|\,\frac{e^2}{|\vec{\boldsymbol{r}}_1 - \vec{\boldsymbol{r}}_2|}\,|\psi_S\rangle \tag{10}$$

$$= -2\int d^3\vec{\boldsymbol{r}}_1\, d^3\vec{\boldsymbol{r}}_2 \psi_a(\vec{\boldsymbol{r}}_1)^* \psi_b(\vec{\boldsymbol{r}}_2)^* \frac{e^2}{|\vec{\boldsymbol{r}}_1 - \vec{\boldsymbol{r}}_2|}\psi_b(\vec{\boldsymbol{r}}_1)\psi_a(\vec{\boldsymbol{r}}_2). \tag{11}$$

(d) The spin-spin interaction can be written as

$$\delta H = \frac{\kappa}{2}\left(|\vec{\boldsymbol{S}}_1 + \vec{\boldsymbol{S}}_2|^2 - |\vec{\boldsymbol{S}}_1|^2 - |\vec{\boldsymbol{S}}_1|^2\right), \tag{12}$$

and recalling that $|\vec{\boldsymbol{S}}_1 + \vec{\boldsymbol{S}}_2|^2\,|s,m\rangle = \hbar^2 s(s+1)\,|s,m\rangle$, the energy difference is

$$E_{s=1} - E_{s=0} = \kappa\hbar^2. \tag{13}$$

So we have to choose

$$\kappa = \frac{\delta E}{\hbar^2} \tag{14}$$

with δE given in Eq. (11).

(e) The energy difference in Eq. (11) has a definite sign. To see this, define

$$F(\vec{\boldsymbol{r}}) = \psi_a(\vec{\boldsymbol{r}}_1)^*\psi_b(\vec{\boldsymbol{r}}_1), \qquad G(\vec{\boldsymbol{r}}_2 - \vec{\boldsymbol{r}}_1) = \frac{1}{|\vec{\boldsymbol{r}}_2 - \vec{\boldsymbol{r}}_1|}, \tag{15}$$

such that

$$\delta E = -2e^2 \int d^3\vec{\boldsymbol{r}}_1\, d^3\vec{\boldsymbol{r}}_2 F(\vec{\boldsymbol{r}}_1) G(\vec{\boldsymbol{r}}_2 - \vec{\boldsymbol{r}}_1) F(\vec{\boldsymbol{r}}_2)^*. \tag{16}$$

Rewriting this expression in terms of the Fourier transform, using

$$G(\vec{\boldsymbol{r}}_2 - \vec{\boldsymbol{r}}_1) = \frac{1}{(2\pi)^3}\int d^3q\, \widetilde{G}(\vec{\boldsymbol{q}})\, e^{i\vec{\boldsymbol{q}}\cdot(\vec{\boldsymbol{r}}_2 - \vec{\boldsymbol{r}}_1)}\ , \tag{17}$$

we find

$$\delta E = -2e^2\frac{1}{(2\pi)^3}\int d^3r_1\, d^3r_2\, d^3q F(\vec{\boldsymbol{r}}_1)\, e^{-i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{r}}_1}\widetilde{G}(\vec{\boldsymbol{q}}) F(\vec{\boldsymbol{r}}_2)^*\, e^{+i\vec{\boldsymbol{q}}_2\cdot\vec{\boldsymbol{r}}_2}\ . \tag{18}$$

We can now perform the integral over r_1 and r_2 using the Fourier transform

$$\widetilde{F}(\vec{\boldsymbol{q}}) = \int d^3r F(\vec{\boldsymbol{r}}) e^{-i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{r}}}, \tag{19}$$

yielding

$$\delta E = -2e^2\frac{1}{(2\pi)^3}\int d^3q \widetilde{F}(\vec{\boldsymbol{q}})\widetilde{G}(\vec{\boldsymbol{q}})\widetilde{F}(\vec{\boldsymbol{q}})^* = -2e^2\frac{1}{(2\pi)^3}\int d^3q \left|\widetilde{F}(\vec{\boldsymbol{q}})\right|^2 \widetilde{G}(\vec{\boldsymbol{q}}). \tag{20}$$

We note that

$$\widetilde{G}(\vec{\boldsymbol{q}}) = \int d^3r e^{-i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{r}}} G(\vec{\boldsymbol{r}}) = \frac{4\pi}{|\vec{\boldsymbol{q}}|^2} \tag{21}$$

is positive definite. Thus the integrand is positive definite, and δE is therefore negative. Using Eq. (14), this implies that $\kappa < 0$, as expected for a ferromagnet.

Quantum Mechanics 2: The Supersymmetric Method

In this problem, you will solve for the energy spectrum of a particle of mass m confined to a potential

$$V(x) = V_0\left[\sec^2\frac{x}{x_0} + \tan^2\frac{x}{x_0}\right] = V_0\left[1 + 2\tan^2\frac{x}{x_0}\right], \tag{1}$$

where $-\frac{\pi}{2}x_0 \le x \le \frac{\pi}{2}x_0$. (Recall that $\sec x = 1/\cos x$, and the equality between the two expressions for $V(x)$ is a consequence of trigonometric identities.) Amazingly, this system is exactly solvable for the special value

$$V_0 = \frac{\hbar^2}{2m}\frac{1}{x_0^2}, \tag{2}$$

and you will derive the spectrum using the "supersymmetric method". (Supersymmetry is a possible symmetry between bosons and fermions, but that fact will not be relevant for this problem.)

(a) (*2 pts*) Consider two Hamiltonians

$$H = A^\dagger A, \qquad \widetilde{H} = AA^\dagger, \tag{3}$$

where A is an unspecified operator. Assume that H has (normalized) eigenstates $|n\rangle$ with

$$H|n\rangle = E_n|n\rangle \qquad (n \ge 1). \tag{4}$$

For every n with $E_n \neq 0$, show that $A|n\rangle$ is an (unnormalized) eigenstate of $\widetilde{H}$. Find the normalized eigenstates $|\widetilde{n}\rangle$ and their eigenvalues $\widetilde{E}_n$ under $\widetilde{H}$. What goes wrong with this argument if $E_n = 0$? Can E_n or $\widetilde{E}_n$ ever be negative? [*Note:* for the remainder of this problem, you can assume that $|\widetilde{n}\rangle$ form a complete basis for $\widetilde{H}$, up to possible zero energy states.]

(b) (*2 pts*) Now consider a specific operator A of the form

$$A = \frac{\partial}{\partial x} + W(x), \tag{5}$$

where $W(x)$ is real. Show that H and $\widetilde{H}$ each describe a particle moving in a potential, in units where $\hbar^2/2m = 1$. Find the two potential energy functions, $V(x)$ and $\widetilde{V}(x)$, corresponding to H and $\widetilde{H}$, respectively.

(c) (*2 pts*) We will be considering Hamiltonians defined on a finite range $-\frac{\pi}{2}x_0 \le x \le \frac{\pi}{2}x_0$. This means that the wave functions will have Dirichlet boundary conditions at $x = \pm\frac{\pi}{2}x_0$, i.e. $\psi(\pm\frac{\pi}{2}x_0) = 0$. Assume that H has a zero energy ground state consistent with these boundary conditions. Find the unnormalized ground state wave function $\psi_0(x)$ for H in terms of $W(x)$. Show that $\widetilde{H}$ *cannot* have a zero energy ground state consistent with these boundary conditions. [*Note:* parts (d) and (e) of this problem can be solved without solving part (c).]

Problem continued on next page.

(d) (*2 pts*) The potential in Eq. (1) is *dual* to a constant potential. That is, there is a $W(x)$ such that for $-\frac{\pi}{2}x_0 \le x \le \frac{\pi}{2}x_0$,

$$V(x) = a, \qquad \widetilde{V}(x) = b\left(\sec^2\frac{x}{x_0} + \tan^2\frac{x}{x_0}\right) = b\left(1 + 2\tan^2\frac{x}{x_0}\right), \tag{6}$$

where a and b are constants. What is $W(x)$, and what are a and b? You may find the following formulas helpful:

$$\int \frac{dx}{1+x^2} = \arctan x, \qquad \int d\theta \sec^2\theta = \tan\theta, \qquad \int d\theta \tan^2\theta = -\theta + \tan\theta. \tag{7}$$

(e) (*2 pts*) Find the energy spectrum and energy eigenstates for the potential in Eq. (1). Does this system have a zero energy ground state? You should assume that all wave functions vanish at $x = \pm\frac{\pi}{2}x_0$ (i.e. Dirichlet boundary conditions), and you should restore all factors of $\hbar$ and m. You do not need to normalize the states.

SOLUTION:

(a) The states $A|n\rangle$ satisfy

$$\widetilde{H}A|n\rangle = AA^\dagger A|n\rangle = AH|n\rangle = AE_n|n\rangle = E_nA|n\rangle\,, \tag{8}$$

so the states $A|n\rangle$ are unnormalized eigenstates of $\widetilde{H}$ with eigenvalue E_n. To normalize the states, note that

$$|A|n\rangle|^2 = E_n. \tag{9}$$

So the non-zero eigenstates of $\widetilde{H}$ are

$$|\widetilde{n}\rangle = \frac{1}{\sqrt{E_n}}A|n\rangle\,, \qquad \widetilde{H}|\widetilde{n}\rangle = E_n|\widetilde{n}\rangle\,, \qquad E_n \neq 0, \tag{10}$$

If $E_n = 0$, then we cannot run through the same argument, because the state $A|n\rangle$ is the zero vector and thus not normalizable. The problem tells us to consider the states $|\widetilde{n}\rangle$ as being a complete set of states up to possible zero energy modes, so we have found all of the non-zero eigenstates of $\widetilde{H}$.

We do know that the eigenvalues of H and $\widetilde{H}$ must be non-negative, since

$$\langle n|H|n\rangle = E_n = |A|n\rangle|^2 \tag{11}$$

is a complete square and thus $E_n \geq 0$. A similar argument holds for $\langle\widetilde{n}|\widetilde{H}|\widetilde{n}\rangle$, giving $\widetilde{E}_n \geq 0$

(b) Using the explicit form for A and recalling that $(\partial/\partial x)^\dagger = -\partial/\partial x$

$$H(x) = A^\dagger A = \left(-\frac{\partial}{\partial x} + W(x)\right)\left(\frac{\partial}{\partial x} + W(x)\right) = -\frac{\partial^2}{\partial x^2} + W(x)^2 - W'(x), \tag{12}$$

$$\widetilde{H}(x) = AA^\dagger = \left(\frac{\partial}{\partial x} + W(x)\right)\left(-\frac{\partial}{\partial x} + W(x)\right) = -\frac{\partial^2}{\partial x^2} + W(x)^2 + W'(x). \tag{13}$$

Indeed, in units with $\hbar^2/2m = 1$, this is one-particle motion in a potential

$$V(x) = W(x)^2 - W'(x), \qquad \widetilde{V}(x) = W(x)^2 + W'(x) \tag{14}$$

(c) If H has a zero energy ground state, then the wavefunction must satisfy

$$A\,\psi_0(x) = 0, \tag{15}$$

which has the solution

$$\psi_0(x) = \exp\left[-\int_0^x dx'\,W(x')\right], \tag{16}$$

where the choice of lower integration limit merely affects the normalization. As long as $W(x)$ goes to infinity fast enough at the Dirichlet boundaries, then we can have a valid, normalizable wave function. To see why $\widetilde{H}$ cannot have zero energy ground state, note that is would have to satisfy

$$A^\dagger \phi_0(x) = 0, \qquad \phi_0(x) = \exp\left[+\int_0^x dx'\, W(x')\right] = \frac{1}{\psi_0(x)}. \tag{17}$$

But if $\psi_0(x)$ satisfied Dirichlet boundary conditions, then $\phi_0(x)$ cannot! So H or $\widetilde{H}$ can have a zero energy ground state, but not both. (Of course, it is consistent for neither H nor $\widetilde{H}$ to have a zero energy ground state. In this situation, we say that supersymmetry is spontaneously broken.)

(d) Recalling that

$$\frac{\partial}{\partial x}\tan x = \sec^2 x, \tag{18}$$

we can see by inspection that

$$W(x) = \frac{1}{x_0}\tan\frac{x}{x_0} \tag{19}$$

works with $-a = b = 1/x_0^2$. Note that we have used the trig identity $\tan^2 x - \sec^2 x = -1$. As a way to derive this more directly, in order for $V(x)$ to be a constant, we need $W' = W^2 + c$ or

$$\frac{dW}{W^2+c} = dx, \tag{20}$$

which is solved by

$$\frac{1}{\sqrt{c}}\arctan\frac{W}{\sqrt{c}} = x - x_1 \qquad \Rightarrow \qquad W = \sqrt{c}\tan(\sqrt{c}(x-x_1)). \tag{21}$$

Plugging this into $\widetilde{V}(x)$, we find $x_1 = 0$ and $\sqrt{c} = \frac{1}{x_0}$.

(e) Because of the Dirichlet boundary conditions at $x = \pm\frac{\pi}{2}x_0$, H is just an infinite square well, offset such that the bottom of the potential is at $-V_0$. The eigenstates and energies for $\widetilde{H}$ are

$$\langle x\,|n\rangle = \sin\left[n\left(\frac{x}{x_0} - \frac{\pi}{2}\right)\right], \qquad E_n = -V_0 + \frac{\hbar^2}{2m}\frac{n^2}{x_0^2} = V_0(n^2-1), \qquad n \geq 1. \tag{22}$$

Here, H does have a zero eigenvalue, so we know that $\widetilde{H}$ cannot have a zero eigenvalue. The eigenvalues for $\widetilde{H}$ are simply

$$\widetilde{E}_n = V_0(n^2-1), \qquad n \geq 2, \tag{23}$$

with a ground state at $3V_0$ (and not at zero). The unnormalized eigenvectors are

$$\langle x\,|\widetilde{n}\rangle = \langle x|\,A\,|n\rangle = \left(\frac{\partial}{\partial x} + \frac{1}{x_0}\tan\frac{x}{x_0}\right)\sin\left[n\left(\frac{x}{x_0}-\frac{\pi}{2}\right)\right] \tag{24}$$

$$= \frac{1}{x_0}\left\{n\cos\left[n\left(\frac{x}{x_0}-\frac{\pi}{2}\right)\right] + \sin\left[n\left(\frac{x}{x_0}-\frac{\pi}{2}\right)\right]\tan\frac{x}{x_0}\right\}. \tag{25}$$

Note that for $n = 1$, this eigenstate is zero, confirming the absence of the zero energy mode for $\widetilde{H}$.