MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS

Academic Programs
Room 4-315

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DOCTORAL GENERAL EXAMINATION

WRITTEN EXAM - CLASSICAL MECHANICS

January 25, 2016

DURATION: 75 MINUTES

1. This examination has two problems. Read both problems carefully before making your choice. Submit ONLY one problem. IF YOU SUBMIT MORE THAN ONE PROBLEM FROM A SECTION, BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.
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3. Calculators may not be used.
4. No books or reference materials may be used.

Classical Mechanics 1: Optimal wind power machine

In this problem you will derive an upper limit on the efficiency with which a machine can harvest kinetic energy from a steady wind.

We model the machine as a flat disk of area A that (through unspecified means) removes kinetic energy and momentum from an airstream flowing perpendicular to the disk. The air flow is steady, laminar, and incompressible, with density ρ, with streamlines that are azimuthally symmetric around the disk's symmetry axis.

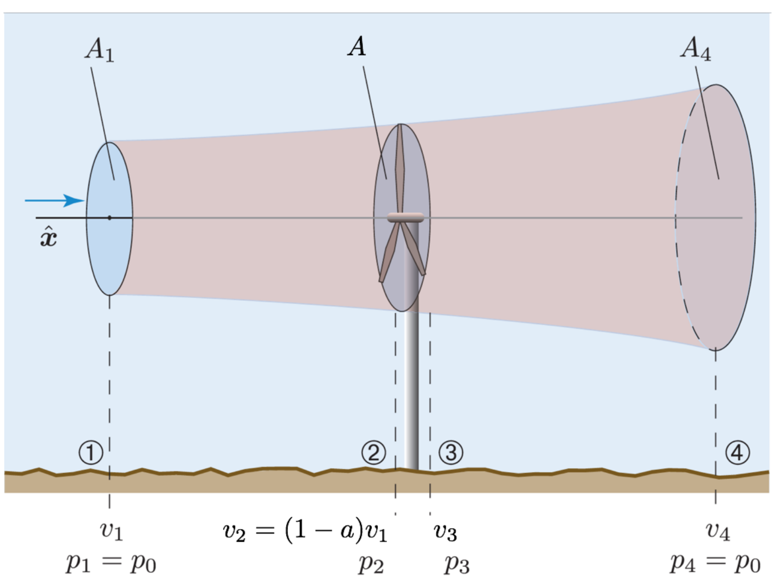
The figure below shows the flow tube that includes the perimeter of the disk. Far upwind, the flow tube has a cross-sectional area $A_1 < A$, the wind speed is v_1, and the air pressure is p_0. Just in front of the turbine, the wind speed has been slowed to $v_2 = (1 - a)v_1$, where $0 < a < 1$ is a parameter characterizing the machine. Just behind the machine, the wind speed is v_3 and the pressure has dropped to p_3. While the machine is operating, $p_2 > p_3 > p_0$. Far downstream, the flow tube has a cross-sectional area $A_4 > A$, the wind speed has slowed further to v_4, the pressure is back to the ambient pressure p_0.

Not all of the quantities defined above are independent. In particular, as you will show, the fraction of wind power extracted by the machine depends solely on a.

(a) (*1 pt*) Use conservation of mass and incompressibility of the flow to relate v_2 and v_3.

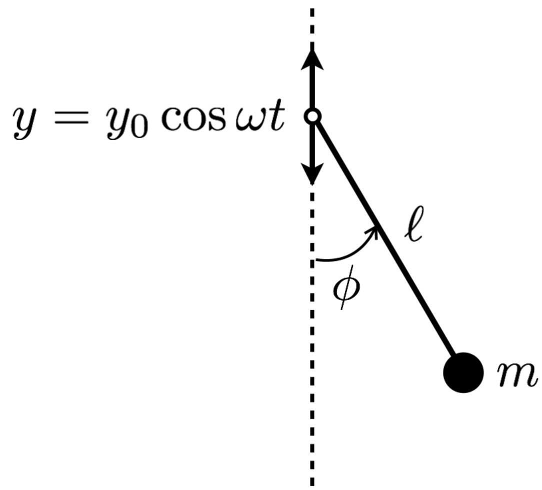
(b) (*7 pts*) Calculate the power P extracted by the disk, in terms of ρ, A, v_1, and a. *Hint:* consider the differences (if any) in the fluxes of mass, momentum and energy into the flow tube at A_1 and out of the flow tube at A_4.

(c) (*2 pts*) Calculate the efficiency $\eta \equiv P/P_0$, where P_0 is the total rate at which kinetic energy of the air would flow through the area A if the turbine were not present. Your answer should be solely a function of a. Then, calculate the maximum possible efficiency $\eta_{\max}$, and value of a for which the maximum is attained.



Classical Mechanics 2: Stabilizing an inverted pendulum

A pendulum has a mass m connected by a rigid rod of length ℓ to a pivot point. An external force causes the pivot point to oscillate vertically, with $y = y_0 \cos \omega t$ and $\omega^2 \gg g/\ell$.



(a) (*1 pt*) Write the Lagrangian for the pendulum, in terms of the angular coordinate ϕ.

(b) (*2 pts*) Derive the equation of motion for ϕ, which by virtue of a cancellation takes the form $\ddot{\phi} = f(\phi, t)$, as opposed to $f(\phi, \dot{\phi}, t)$.

Next we will find an approximate solution of the form $\phi = \phi_s + \phi_f$. The "slow" component $\phi_s(t)$ is the solution of the equation of motion after time-averaging over the rapid oscillations of the pivot point. The "fast" component $\phi_f(t)$ is a small perturbation ($|\phi_f(t)| \ll 1$), and averages to zero over the rapid oscillations.

(c) (*5 pts*) Derive a leading-order equation for ϕ_f. Solve it, and then use the solution to derive a time-averaged equation for ϕ_s of the form

$$\ddot{\phi}_s = A \sin \phi_s + B \sin \phi_s \cos \phi_s, \tag{1}$$

where A and B depend only on the constants g, ℓ, ω, and y_0.

(d) (*2 pts*) Rewrite the preceding equation for ϕ_s in the form $\ddot{\phi}_s = -dU_{\text{eff}}/d\phi_s$, i.e., motion in an effective potential. Then derive a condition on ω and y_0 such that $\phi_s = \pi$ is a stable equilibrium point.

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DOCTORAL GENERAL EXAMINATION

WRITTEN EXAM - ELECTRICITY AND MAGNETISM

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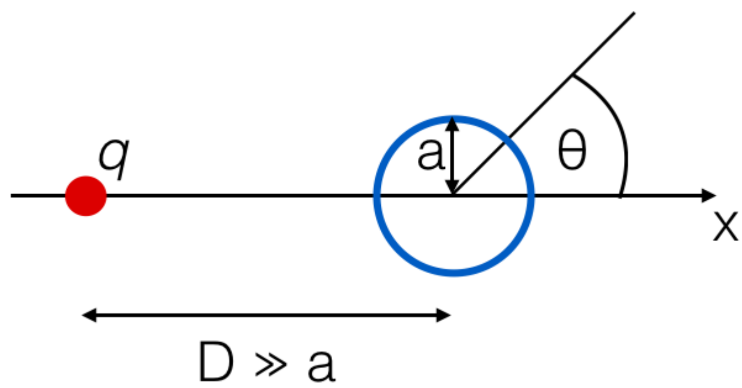
Electromagnetism 1: Induced Dipole Moment on Conducting Sphere

A positive electric point charge $+q$ is placed at a distance D from a conducting sphere of radius a that carries no net charge ($\mathrm{D} \gg a$).



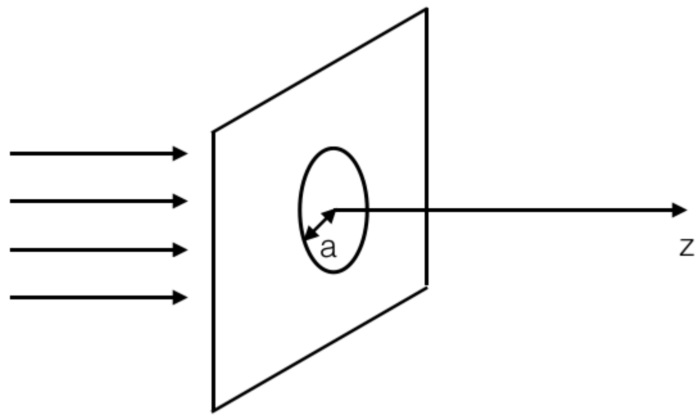
(a) (*4 pts*) Calculate the induced dipole moment of the sphere. In which direction does it point?

(b) (*3 pts*) Determine the induced surface charge density $\sigma(\theta)$ (see figure).

(c) (*3 pts*) Determine the force of the sphere (magnitude and direction).

Electromagnetism 2: Light intensity behind circular hole

A circular hole of radius a, cut in a large opaque screen, is illuminated at normal incidence with plane monochromatic radiation of wavelength λ (see figure). You may treat the radiation field as a scalar.



(a) (*5 pts*) Calculate the light intensity at a point P lying on the symmetry axis of the aperture as a function of the distance from the aperture z. Assume that $z \gg a$, but that z is comparable to a^2/λ.

(b) (*2 pts*) Normalize the light intensity so that, in the absence of a screen and aperture, the intensity is 1. With the screen and aperture in place, and for $a^2/(4\lambda) \le z \le 2a^2/\lambda$, what is the maximum intensity and at what value(s) of z does it occur?

(c) (*3 pts*) What would be the light intensity distribution on the symmetry axis along z (in the same region as in (a)) if the screen plus hole were replaced by a opaque disk of radius a?

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WRITTEN EXAM - QUANTUM MECHANICS

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Quantum Mechanics 1: Harmonic Oscillator Gains Weight over the Holidays

Consider a particle moving in 1-d subject to a harmonic potential, and suppose that the mass is slowly increasing; i.e.

$$H(t) = \frac{p^2}{2m(t)} + \frac{1}{2}m(t)\omega^2 x^2, \tag{1}$$

where $m(t) = m_0 e^{\nu t}$ for some constants m_0, ν. Suppose that at time $t = 0$ the particle is in the ground state of $H(0)$.

(a) (*3 pts*) For small values of ν, the adiabatic theorem guarantees that the particle will remain approximately in the ground state at later times. How small does ν have to be to make this true?

(b) (*2 pts*) Suppose that there exists an operator A satisfying

$$[A, p] = \frac{i\nu p}{2} \qquad \text{and} \qquad [A, x] = \frac{-i\nu x}{2}. \tag{2}$$

Show that this implies that

$$H(t) = e^{iAt} H(0) e^{-iAt}. \tag{3}$$

(c) (*2 pts*) Find a Hermitian operator A satisfying Eq. (**??**). *[Hint: consider products of x and p.]*

(d) (*3 pts*) Let $|\psi(t)\rangle$ be the solution of the Schrödinger equation. It turns out that

$$|\psi(t)\rangle = e^{-iAt} e^{iBt} |\psi(0)\rangle \tag{4}$$

Find B.

Quantum Mechanics 2: Scattering from a Spherical Potential

A spinless non-relativistic particle with momentum $\hbar\mathbf{k}$, represented by a normalized plane wave state $|\mathbf{k}\rangle$, is incident on a region with a (weak) potential $V(r)$. An outgoing scattered particle with momentum $\hbar\mathbf{k}'$ is represented by the plane wave state $|\mathbf{k}'\rangle$. If we observe outgoing particles in a small solid angle $d\Omega$ about the direction of $\mathbf{k}'$, Fermi's golden rule tells us that the transition rate $\omega_{kk'}$ is given by

$$\omega_{kk'} = \frac{2\pi}{\hbar} g(E_{k'}) | \langle \mathbf{k}' | V | \mathbf{k} \rangle |^2, \tag{1}$$

where $g(E)$ is the density of (outgoing) states defined from $dN = g(E_k) dE_{k'}$, where dN is the number of states with energy in the interval dE_k about E_k and direction within $d\Omega$. For normalization work in a large cubic box of volume L^3.

(a) (*2 pts*) Write the normalized wave function $\langle \mathbf{x} | \mathbf{k} \rangle$ for a plane wave state. Calculate $g(E_k)$ in terms of $m, L, k', d\Omega$, and $\hbar$.

(b) (*2 pts*) Compute the probability current density $\mathbf{J}_{\text{inc}}$ corresponding to the incident state $|\mathbf{k}\rangle$ [recall: $\mathbf{J} = \frac{\hbar}{m} \text{Im}(\psi^* \nabla \psi)$]. What are the units of J_{inc}? What are the units of $\omega_{kk'}$?

(c) (*2 pts*) Relate $\mathbf{J}_{\text{inc}}$, the differential cross section $d\sigma$, and $\omega_{kk'}$ to obtain a formula for $\frac{d\sigma}{\Omega}$ of the form

$$\frac{d\sigma}{\Omega} = f(m, L, \hbar, k, k') | \langle \mathbf{k}' | V | \mathbf{k} \rangle |^2. \tag{2}$$

Give the explicit form of f. Since the scattering is elastic, $k = k'$ may be used to simplify the answer.

(d) (*4 pts*) Consider the potential $V(r)$ equal to V_0 for $r < a$ and zero for $r > a$. Let $q \equiv k' - k$ and $q = |q|$. Give q in terms of k and the deflection angle θ? Calculate $\langle k' | V | k \rangle$ as a function of q, a, and other constants. Find the differential cross section $\frac{d\sigma}{\Omega}$ and its approximate value when $qa \ll 1$.

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Statistical Mechanics 1: Leaky spacesuit

Suppose that Matt Damon is floating in space, wearing a spacesuit that (for simplicity) contains pure nitrogen gas. The spacesuit also contains a small hole which we assume is much smaller than the mean free path of the gas. Assume further that the escape rate is slow enough that the remaining gas remains in equilibrium by collisions among the gas molecules; neglect any interaction between the gas and Matt Damon. Neglect also the thickness of the spacesuit.

(a) (*1 pt*) What is the average energy of a gas molecule inside the spacesuit at temperature T? Treat the rotational modes of the diatomic gas classically, but neglect the vibrational mode which is not excited at room temperature.

(b) (*4 pts*) What is the average energy, including both kinetic and rotational, per molecule of gas that escapes when the gas inside the spacesuit is at temperature T?

(c) (*2 pts*) Suppose the spacesuit initially contains N_0 molecules at temperature T_0. Once there are only N molecules remaining, what is the temperature T of the remaining gas?

(d) (*3 pts*) Eventually all of the gas will escape. Assume that Matt Damon does not rotate so that the hole is always pointing in the same direction. What will his momentum be once all the gas is gone ($N = 0$)?

For convenience we include the following formula.

$$\int_0^\infty e^{-u} u^n du = \Gamma(n+1) = \begin{cases} n! & \text{if } n \text{ is an integer} \\ \frac{(2n+1)!}{4^{n+1}(n+1/2)!}\sqrt{4\pi} & \text{if } n+1/2 \text{ is an integer} \end{cases} \tag{1}$$

Statistical Mechanics 2: Atom-Molecule Equilibrium in a Harmonic Trap

In this problem we are analyzing the equilibrium between atoms and molecules trapped in three-dimensional, spherically symmetric harmonic potentials. There are two species of atoms A, B with masses m_A, m_B. Atoms A and B can form a molecule M in a reaction $A + B \leftrightarrow M$, where the molecular binding energy is $-E$ with $E > 0$, i.e., a molecule at rest has less energy than a pair of atoms A, B at rest. We assume that two atoms of the same species do not form molecules, and that there is only one relevant state of the molecule (rotational and vibrational modes are frozen out). The numbers of particles of each species are denoted by N_A, N_B, and N_M. The system is assumed to be in thermal equilibrium at sufficiently high temperature T, such that the fermionic or bosonic character of the particles is irrelevant.

(a) (*1 pt*) For each particle, the eigenenergies are given by $E_j = (j + 3/2)\hbar\omega$, with degeneracy g_j, where the same value of ω applies independently of whether the particles are atoms A or B, or molecule M. What is the degeneracy g_j for the energy E_J? If you do not know how to compute g_j, you may simply leave it in symbolic form and go on to do the rest of the problem.

(b) (*2 pts*) Determine the single-particle partition functions Z_α, where $\alpha = A, B, M$. Given particle numbers N_α, what is the condition on Z_α and N_α that allows one to ignore the fermionic or bosonic character of the particles?

(c) (*2 pts*) Find the partition function for the complete system of atoms and molecules at fixed particle numbers N_A, N_B, and N_M expressed in terms of the single-particle atomic partition function Z_A.

(d) (*5 pts*) Assume that initially there are N_{A_0} and N_{B_0} atoms A, B and no molecules. The system then reaches atom-molecule equilibrium at temperature T with particle number N_α, with $\alpha = A, B, M$. Determine the equilibrium molecule number N_M in terms of N_A, N_B, the single-particle partition functions, and the other quantities given in the problem. Explain why the conversion efficiency of atom pairs into molecules remains small for $E \sim kT$ even though the molecules have less energy. Use Stirling's approximation: $\ln N! \approx N \ln N - N$ to simplify the calculations.