MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS

Academic Programs
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DOCTORAL GENERAL EXAMINATION

WRITTEN EXAM

January 25–28, 2021

DURATION: 75 MINUTES PER SECTION

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DOCTORAL GENERAL EXAMINATION

WRITTEN EXAM - CLASSICAL MECHANICS

Monday, January 25th, 2021

DURATION: 75 MINUTES

1. This examination has two problems. Read both problems carefully before making your choice. Submit ONLY one problem. IF YOU SUBMIT MORE THAN ONE PROBLEM , BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.
2. Computers are only to be used for the zoom session and downloading/uploading the exam. Calculators may not be used.
3. No books or reference materials (online or offline) may be used.

Classical Mechanics 1: A relativistic pendulum

[Note: in this problem we do **not** assume that angles are small]

To take a break from their usual work, a tenacious phD student has decided they need to calculate relativistic corrections to the motion of a simple pendulum, and verify they are indeed negligible. The pendulum is constituted by a bob of mass m at the end of an inextensible wire of length ℓ and zero mass. The other end of the wire is fixed to a fulcrum. Unless otherwise stated, all quantities are defined in a frame where the fulcrum of the pendulum is at rest ("Lab" frame).

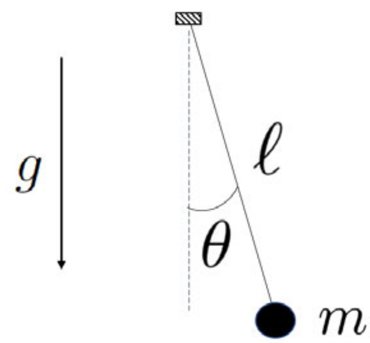

Figure 1: An ambitious project.

A uniform gravitational field exists, characterized by a magnitude g. In order to keep mathematics simpler, we assume that gravity can be treated as a force proportional to the "relativistic mass" of the particle and we take the force on the particle to be:

$$|F| = \frac{mg}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where v is the norm of the velocity vector.

a) [0.5 pt] Write an expression for the differential of the particle's potential – $\mathrm{d}V$ – as its position varies by $\mathrm{d}\theta$. Use polar coordinates and set the origin at the fulcrum (See Fig 1).

b) [1 pt] Remembering that the Lagrangian of a *free* relativistic particle of mass m and velocity $\vec{v}$ is

$$L(v) = -mc^2\sqrt{1 - \frac{v^2}{c^2}},$$

write down the Lagrangian and the Hamiltonian of the relativistic pendulum.

[Note: for the potential, you might use the appropriate integral of what you found in part a), without solving it]

c) [1 pt] Assume the bob is released from an initial angle θ_0 with zero velocity. Write an expression for the gamma factor of the bob at a generic angle θ as function of the potential, the value of the potential at θ_0, and constants.

d) [2 pt] Write an expression for the potential that does **not** contain integrals or derivatives. Your answer must have the form:

$$V(\theta) = h(\theta) + K$$

where $h(\theta)$ and K are a function and a constant, respectively.

e) [1 pt] Write the velocity of the bob at position θ. You answer must **not** contain the potential.

f) [1.5 pt] Write the equation of motion for θ. You answer must **not** contain integrals or derivatives.

[Note: you do not need to solve the equation]

g) [1 pt] Using only Newtonian mechanics, find an integral expression for the **Newtonian** period of the pendulum.

h) [2 pt] Calculate the period of the relativistic pendulum and show that you recover the Newtonian result in the appropriate limit. Explicitly write down the leading order relativistic correction. Is the relativistic period smaller or larger than the Newtonian one?

[Note: Your answer will contain an integral, you do not need to solve it]

- Notable 4-vectors, for a particle of mass m with velocity $\vec{u}$, in cartesian coordinates:

  4-position: (t, x, y, z).
  4-velocity: $(\gamma(u), \gamma(u)u^x, \gamma(u)u^y, \gamma(u)u^z)$.
  4-momentum: $m \times 4 -$ velocity.

- Gamma factor: $\gamma^2(u) \equiv 1/(1 - u^2/c^2)$.

Classical Mechanics 2: The precession of the perihelion

A particle moves in a region described by the central potential

$$V(r) \equiv -\frac{k}{r}e^{-r/a}$$

with $k > 0$ and $a > 0$. We will work in the center-of-mass frame, and call μ the reduced mass of the system.

a) [0.5 pt] Write the Lagrangian of the system. You can assume planar motion and call ϕ your angular variable;

b) [1 pt] Write the equations of motion;

c) [1 pt] Find an equation for r of the form

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \tau\left[E - V_{\text{eff}}(r)\right]$$

Where E is the total energy of the system, $V_{\text{eff}}(r)$ an effective potential and τ a real constant that you must express in terms of known variables;

d) [2.5 pt] Show that there exist a maximum angular momentum $\ell_{\max}$ above which no bound orbits are possible. Give an expression for $\ell_{\max}$ in terms of μ,k, a and numerical constants. Your answer must be explicit and must not contain exponential functions.

e) [2 pt] Find a differential equation for $u(\phi)$, where $u \equiv 1/r$;

Consider now a situation where $\ell < \ell_{\max}$ so that a stable circular orbit is possible at $r = r_0$. We will focus on an orbit which is very close to this one, i.e. $r(t) = r_0 + \delta r(t)$ with $|\delta r(t)| \ll r_0$.

f) [3 pt] Assuming $r_0 \ll a$, find an approximate expression for the advance of the perihelion in each revolution. Your solution must be on the form

$$\Delta\theta = \frac{A}{r_0^2} + \frac{B}{r_0} + Cr_0 + Dr_0^2$$

Where A and B, C and D are real constants you must calculate (some or all of them might be zero) [Note: your answer *cannot* depend on k, μ or ℓ].

– Potentially useful data:

$$\sqrt{2} = 1.4, \sqrt{3} = 1.7, \sqrt{5} = 2.2, \sqrt{7} = 2.6$$
$$e^{-3.4} = 0.03, e^{-1.6} = 0.2, e^{-0.8} = 0.45, e^{-0.1} = 0.90$$

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DOCTORAL GENERAL EXAMINATION

WRITTEN EXAM - ELECTRICITY AND MAGNETISM

Tuesday, January 26th, 2021

DURATION: 75 MINUTES

1. This examination has two problems. Read both problems carefully before making your choice. Submit ONLY one problem. IF YOU SUBMIT MORE THAN ONE PROBLEM , BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.
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Electromagnetism 1: Parallel plates with an indent

Consider two parallel infinite conducting plates separated along the z direction by a distance d. The bottom plate is grounded and the upper plate is held at a potential such that there is a uniform electric field $E_0\hat{z}$ between the plates. A hammer is struck against the bottom plate from below, leaving a hemispherical indent of radius $a \ll d$ (still held at zero potential) protruding towards the top plate.

(a) By considering suitable image charges, find the potential field between the plates $\Phi(r, \theta)$ in polar coordinates centered on the middle of the hemisphere. To find the potential, consider the following questions:

  i) (1 pts) What arrangement of image charges would produce the uniform field between the plates without the indent present?

  ii) (2 pts) Now with the indent present, what image charges (that is, their magnitude and position) must be added to keep the indent at zero potential?

  iii) (1 pts) What is the potential resulting from all of these charges?

  iv) (2 pts) By enforcing that the field far from the indent, $a \ll r \ll d$, is not affected by the indent, and thus takes the original uniform form, determine the values of the image charges in terms of E_0 and geometric parameters.

(b) (4 pts) Fnd the total charge induced on the hemispherical indent in the limit $a \ll d$. [Hint: Note that $\int_0^{\pi/2} d\theta \sin\theta \cos\theta = 1/2$.]

Electromagnetism 2: Electromagnetic angular momentum

An infinitely long wire is on the z-axis and has a charge per unit length $-\lambda$. A plastic cylindrical shell of radius R is concentric about the z-axis and has a charge per unit area $\sigma = +\lambda/2\pi R$ uniformly distributed over its surface (and fixed to that surface). The cylindrical shell is suspended so that it can rotate freely about the z-axis without friction.

For $t < 0$, the shell is initially at rest and immersed in a constant magnetic field $\vec{B}_{ext} = B_0\hat{z}$ produced by external currents on a concentric solenoid of radius $R_S \gg R$.

(a) (2 pts) What is the initial (i.e., $t < 0$) electric field for all $\rho < R_S$ (where $\vec{\rho}$ is the cylindrical radius vector)?

(b) (2 pts) The electromagnetic angular momentum density is given by

$$\vec{\mathcal{L}} = \frac{1}{4\pi c}\vec{\rho} \times (\vec{E} \times \vec{B}). \tag{1}$$

Calculate this and find the total electromagnetic angular momentum per unit length along the z-axis for the initial configuration. Give your answer in terms of λ, B_0, R, and the speed of light, c.

(c) (2 pts) Beginning at $t = 0$, the external current in the solenoid is slowly reduced to zero over some time $t_0 \gg R/c$, so that $\vec{B}_{ext} = \hat{z}B_{ext}(t)$ with $B_{ext}(0) = B_0$ and $B_{ext}(t_0) = 0$. The shell will begin to rotate (assume that the shell is so massive that it rotates very slowly, and any self-magnetic field generated by its rotation may be neglected). Use the torque generated by the induced electric field to calculate the angular momentum per unit length of the shell at time $t = t_0$. How does this compare to the electromagnetic angular momentum at $t = 0$?

(d) (4 pts) The flux density of electromagnetic angular momentum across a surface whose normal is $\hat{\rho}$ is $\mathcal{F} = -(\vec{\rho}\times \overleftrightarrow{T}) \cdot \hat{\rho}$, where $\overleftrightarrow{T}$ is the Maxwell stress tensor, given by

$$\overleftrightarrow{T} = \frac{1}{4\pi}\left[\vec{E}\vec{E} + \vec{B}\vec{B} - \frac{1}{2}\overleftrightarrow{I}\left(E^2 + B^2\right)\right], \tag{2}$$

where $\overleftrightarrow{I}$ is the identity matrix. Use this to calculate the total electromagnetic angular momentum per unit length that flows across an imaginary cylindrical shell at $\rho = R-\epsilon$ from $t = 0$ to $t = t_0$ ($\epsilon \ll 1$). How does this compare to your answer in (c)?

Note, Eq. (1) and (2) are in Gaussian units. Their forms in SI units if you prefer to work with them are

$$\vec{\mathcal{L}} = \epsilon_0\, \vec{\rho} \times (\vec{E} \times \vec{B})$$

, and

$$\overleftrightarrow{T} = \epsilon_0\left[\vec{E}\vec{E} + c^2\vec{B}\vec{B} - \frac{1}{2}\overleftrightarrow{I}\left(E^2 + c^2B^2\right)\right].$$

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DOCTORAL GENERAL EXAMINATION

WRITTEN EXAM - STATISTICAL MECHANICS

Thursday, January 28th, 2021

DURATION: 75 MINUTES

1. This examination has two problems. Read both problems carefully before making your choice. Submit ONLY one problem. IF YOU SUBMIT MORE THAN ONE PROBLEM , BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.
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Statistical Mechanics 1: Magnetic Properties

Consider a solid containing N electrons localized on lattice sites. Let each electron have magnetic moment $\vec{\mu} = g\mu_0\vec{s}$, where $g = 2$, μ_0 is the Bohr magneton, and $\vec{s}$ the spin angular momentum.

(a) Suppose that each spin is subject to an effective magnetic field H_{eff}. Show that at thermal equilibrium the total magnetic moment M of the solid is given by

$$M = N\mu_0 \tanh\left(\frac{\mu_0 H_{eff}}{k_B T}\right),$$

where k_B is the Boltzmann constant and T is the temperature. (*2 pts*)

(b) In the case of a ferromagnetic solid, the effective field H_{eff} acting on a given spin consists of the externally applied field H and a local "Weiss field" $H_{loc} = qM$ proportional to the alignment of the near neighbor spins. Write down an equation which determines the magnetization M in a self consistent manner. (*1 pt*)

(c) Show that below a certain critical temperature T_C there is a spontaneous alignment of the spins, and obtain an expression for T_C in terms of the constants defined above. (*2 pts*)

(d) Show that in the vicinity of the critical temperature T_C, but for $T < T_C$ that the magnetization increases with temperature as

$$\frac{M}{N\mu_0} \sim \sqrt{3}\left(\frac{T_C - T}{T_C}\right)^{1/2}$$

(*3 pts*)

(e) Show that the susceptibility of the solid $\lim_{H\to 0}(\partial M/\partial H) = \chi$ diverges as $T \to T_C$ from above in accordance with the Curie-Weiss Law

$$\chi \cong \frac{N\mu_0^2}{k_B(T - T_C)}$$

(*2 pts*)

Potentially useful expansion:

$\tanh(x) \approx x - \frac{x^3}{3}$ for $x \ll 1$.

Statistical Mechanics 2: Electron and Helium Gases

A free-sliding, heat-conducting piston divides a cylinder into two compartments labeled 1 and 2, respectively. An electron gas is placed in compartment 1, and a helium gas in compartment 2, with respective particle number densities n_1 and n_2. The entire system is kept at a uniform temperature T, such that the electron gas is effectively at absolute zero, but the helium gas is at such a high temperature that it may be treated classically.

(a) Find the equilibrium relation among n_1, n_2, and T. (*4 pts*)

(b) Find the condition on n_1/n_2 such that the electron gas is effectively at absolute zero. (*2 pts*)

(c) Find the condition on n_1/n_2 such that the helium gas can be treated classically. (*2 pts*)

(d) Give a numerical example to show that it is possible to satisfy both of the above conditions simultaneously. (*2 pts*)

Potentially useful physical constants:

electron mass $m_e = 9.1 \times 10^{-31}$ kg

proton mass $m_p = 1.67 \times 10^{-27}$ kg

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DOCTORAL GENERAL EXAMINATION

WRITTEN EXAM - QUANTUM MECHANICS

Wednesday, January 27th, 2021

DURATION: 75 MINUTES

1. This examination has two problems. Read both problems carefully before making your choice. Submit ONLY one problem. IF YOU SUBMIT MORE THAN ONE PROBLEM , BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.
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Quantum Mechanics 1: τ-atoms

The third generation τ lepton is a heavy version of the electron with mass $m_\tau c^2 = 1.777$ GeV (compared with the electron mass of $m_e c^2 = 511$ keV). All of its other properties (charge, spin, g-factor etc.) are identical to those of the electron.

In this question we will study the properties of heavy 'Tauonic' atoms. Specifically we will consider the binding of a single τ to a uranium nucleus ($Z = 92$) under the coulomb interaction (assume no other leptons, i.e. τ or e^-, are bound to the nucleus). For simplicity we will assume that the Uranium nucleus is infinitely heavy with a uniform spherical charge distribution of radius $R = 7.5$ fm that the lepton can pass through.

(a) (2 pt) Starting from the electric field due to the nucleus, derive the Hamiltonian describing the motion of the τ in the coulomb field of the nucleus. Note: you should treat the tau non-relativistically and will want to consider separately the case when the τ is inside and outside the nucleus. You can also ignore spin-dependent forces.

(b) (3 pt) Assuming that the τ is confined entirely inside the nucleus, find the general expression for all of its energy levels and determine the angular momentum of the ground state and first excited state. Hint: can the relevant Hamiltonian be expressed as that of a harmonic oscillator?

(c) (2.5 pt) Calculate the first order correction to the ground state energy due to relativistic corrections to the τ kinetic energy. Remember that this is a *three* dimensional problem. Hint: note that the relativistic kinetic energy is given by $T = \sqrt{\vec{p}^2c^2 + m^2c^4} - mc^2 \approx \frac{\vec{p}^2}{2m} - \frac{1}{8}\frac{(\vec{p}^2)^2}{m^3c^2}$. You can ignore the finite size of the nucleus in this part of the problem also.

(d) (2.5 pt) Derive an expression for the first order correction to the ground state energy due to the finite size of the nucleus in terms of a single definite integral. Note: you do not need to perform the final integration, but make sure to define everything in your expression. You can ignore relativistic effects in this part of the problem.

Quantum Mechanics 2: non-relativistic inelastic scattering on a two level atom

Consider a scattering problem for a particle in 1D with 2 spin states that is described by the Hamiltonian $H = \frac{p^2}{2m} + \frac{c}{2}\sigma_z + u\sigma_x\delta(x)$, where c and u are arbitrary positive constants. The particle, incoming from $x = -\infty$, is prepared in the state corresponding to the lower of the two energies. The kinetic energy of the incident wave from $x < 0$ is E_v.

(a) (2 pt) Assuming $E_v > c$, express the "free wave" (i.e. away from $x = 0$) solutions of the Schrodinger equation in the scattering channels in terms of reflection and transmission amplitudes.

(b) (2 pt) Write down the matching equation for the wave function $\psi_\sigma(x)$ and its derivative at $x = 0$, where σ is the spin index.

(c) (3 pt) Find the total transmission coefficient T in terms of the free wave momenta.

(d) (3 pt) Repeat (a) and (c) for the case $E_v < c$.