Classical Mechanics Spring 2021

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Classical Mechanics 1: A relativistic pendulum

*[Note: In this problem we do **not** assume that angles are small]*

To take a break from their usual work, a tenacious phD student has decided they need to calculate relativistic corrections to the motion of a simple pendulum, and verify they are indeed negligible. The pendulum is consistituted by a bob of mass m at the end of an inextensible wire of length l and zero mass. The other end of the wire is fixed to a fulcrum. Unless otherwise states, all quantities are defined in a frame where the fulcrum of the pendulum is at rest ("Lab" frame).

A uniform gravitational field exists, characterized by a magnitude g. In order to keep mathematics simpler, we assume that gravity can be treated as a force proportional to the "relativistic mass" of the particle and we take the force on the particle to be:

$$|F| = \frac{mg}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{1}$$

where v is the norm of the velocity vector.

a) [0.5 pt] Write an expression for the differential of the particle's potential — dV — as its position varies by $d\theta$. Use polar coordinates and set the origin at the fulcrum (See Fig. 1).

$$\mathrm{d}V = \gamma mgl\sin\theta \mathrm{d}\theta = -\gamma mgl\, \mathrm{d}(\cos\theta) \tag{2}$$

b) [1 pt] Remembering that the Lagrangian of a free relativistic particle of mass m and velocity $\vec{v}$ is

$$L(v) = -mc^2\sqrt{1-\frac{v^2}{c^2}}, \tag{3}$$

write down the Lagrangian and the Hamiltonian of the relativistic pendulum.

[Note: for the potential, you might use the appropriate integral of what you found in part a), without solving it]

$$L(\theta, v) = -\frac{mc^2}{\gamma} - mgl\int_{\theta_0}^{\theta}\gamma(\theta')\sin\theta' \mathrm{d}\theta' = -\frac{mc^2}{\gamma} + mgl\int_{\cos\theta_0}^{\cos\theta}\gamma(\cos\theta')\mathrm{d}(\cos\theta') \tag{4}$$

To find the Hamiltonian,

$$p = \frac{\partial L}{\partial v} = \gamma mv \tag{5}$$

$$H = pv - L = \gamma mc^2 + V = \gamma mc^2 - mgl\int_{\cos\theta_0}^{\cos\theta}\gamma(\cos\theta')\mathrm{d}(\cos\theta') \tag{6}$$

Note: In Eq. (5), we do NOT differentiate the γ in the integral expression of V. The justification is not quite clear to me, but the alternative is a road filled with pain.

c)[1 pt] Assume the bob is released from an initial angle θ_0 with zero velocity. Write an expression for the gamma factor of the bob at a generic angle θ as function of the potential, the value of the potential at θ_0, and constants.

Using the fact that $\dot{H} = 0$ (conservation of energy),

$$\gamma mc^2 + V(\theta) = mc^2 + V(\theta_0) \tag{7}$$

*d) [2 pt] Write an expression for the potential that does **not** contain integrals or derivatives. Your answer must have the form:*

$$V(\theta) = h(\theta) + K \tag{8}$$

where $h(\theta)$ and K are a function and constant, respectively.

From Eq. (7), $mc^2\gamma(\theta_1) + V(\theta_1) = mc^2\gamma(\theta_2) + V(\theta_2)$ for any θ_1, θ_2. Therefore $mc^2\gamma'(\theta) = -V'(\theta)$, or more conveniently:

$$mc^2 \frac{d\gamma}{d\cos\theta} = -\frac{dV}{d\cos\theta} = mgl\gamma(\cos\theta) \tag{9}$$

$$\Rightarrow \gamma(\cos\theta) = \exp\left[\frac{gl}{c^2}\left(\cos\theta - \cos\theta_0\right)\right] \tag{10}$$

Using Eq. (7) again,

$$V(\theta) = V(\theta_0) - mc^2\left(\exp\left[\frac{gl}{c^2}\left(\cos\theta - \cos\theta_0\right)\right] - 1\right) \tag{11}$$

*e) [1 pt] Write the velocity of the bob at position θ. You answer must **not** contain the potential.*
From Eq. 10,

$$\gamma(\cos\theta) = \exp\left[\frac{gl}{c^2}\left(\cos\theta - \cos\theta_0\right)\right] \tag{12}$$

$$\Rightarrow v(\theta) = c\sqrt{1 - \exp\left[2\frac{gl}{c^2}\left(\cos\theta_0 - \cos\theta\right)\right]} \tag{13}$$

*f) [1.5 pt] Write the equation of motion for θ. You answer must **not** contain integrals or derivatives.*
[Note: You do not need to solve the equation]
Our answer does contain one derivative, $\dot{\theta}$. I will assume this is acceptable.

$$v(\theta) = l\dot{\theta} = c\sqrt{1 - \exp\left[2\frac{gl}{c^2}\left(\cos\theta_0 - \cos\theta\right)\right]} \tag{14}$$

Note: I am not sure if this is the intended answer: it is extremely short relative to the allocated points.
*g) [1 pt] Using only Newtonian mechanics, find an integral expression for the **Newtonian** period of the pendulum.*
Using conservation of energy:

$$\frac{1}{2}ml^2\dot{\theta}^2 = mgl(\cos\theta - \cos\theta_0) \tag{15}$$

$$\mp\sqrt{\frac{l}{2g}}\int \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \int dt \tag{16}$$

$$\Rightarrow T = 4\sqrt{\frac{l}{2g}}\int_0^{\theta_0} \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_0}} \tag{17}$$

The sign depends on the quadrant of the oscillation, just choose it such that $T > 0$.
h) [2 pt] Calculate the period of the relativistic pendulum and show that you recover the Newtonian result in the appropriate limit. Explicitly write down the leading order relativistic correction. Is the relativistic period smaller or larger than the Newtonian one?
[Note: Your answer will contain an integral, you do not need to solve it]
Analogously, from Eq. (14):

$$T = 4\frac{c}{l}\int_0^{\theta_0} \frac{\mathrm{d}\theta}{\sqrt{1 - \exp\left[2\frac{gl}{c^2}\left(\cos\theta_0 - \cos\theta\right)\right]}} \approx 4\frac{c}{l}\int_0^{\theta_0} \frac{\mathrm{d}\theta}{\sqrt{2\frac{gl}{c^2}\left(\cos\theta - \cos\theta_0\right) - \frac{1}{2}\left(2\frac{gl}{c^2}\left(\cos\theta - \cos\theta_0\right)\right)^2}} \tag{18}$$

$$\approx 4\sqrt{\frac{l}{2g}}\int_0^{\theta_0} \mathrm{d}\theta\left(\frac{1}{\sqrt{\cos\theta - \cos\theta_0}} + \frac{gl}{2c^2}\sqrt{\cos\theta - \cos\theta_0}\right) \tag{19}$$

The second term in Eq. (19) is always positive, therefore the relativistic period will be longer than the Newtonian one (as expected).
Note: After preparation of this note I found the reference which this question was based on [1], with consistent answers.
Note 2: The most important steps in this problem are Eqs. (7) and (10), in particular turning (7) into a differential equation to simplify the integral expression for V.

Classical Mechanics 2: The precession of the perihelion

A particle moves in a region described by the central potential

$$V(r) = -\frac{k}{r}e^{-r/a} \tag{20}$$

with $k > 0$ and $a > 0$. We will work in the center-of-mass frame, and call μ the reduced mass of the system.

a) [0.5 pt] Write the Lagrangian of the system. You can assume planar motion and call ϕ your angular variable;

$$\mathcal{L} = \frac{1}{2}\mu\dot{r}^2 + r^2\dot{\phi}^2 - V(r) \tag{21}$$

b) [1 pt] Write the equations of motion;

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = u\ddot{r} = \frac{\partial \mathcal{L}}{\partial r} = -V'(r) + \mu r\dot{\theta}^2 = -V'(r) + \frac{L^2}{\mu r^3} \tag{22}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{d}{dt}(\mu r^2\dot{\phi}) = \frac{\partial \mathcal{L}}{\partial \phi} = 0 \Rightarrow \mu r^2\dot{\phi} = L \tag{23}$$

c) [1 pt] Find an equation for r of the form

$$\left(\frac{dr}{dt}\right)^2 = \tau[E - V_{\text{eff}}(r)] \tag{24}$$

Where E is the total energy of the system, $V_{eff}(r)$ an effective potential and τ a real constant that you must express in terms of known variables;

From conservation of energy:

$$\frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r) = E \tag{25}$$

$$\left(\frac{dr}{dt}\right)^2 = \frac{2}{\mu}\left[E - \frac{L^2}{2\mu r^2} - V(r)\right] \tag{26}$$

So $\tau = 2/\mu$ and $V_{\text{eff}}(r) = V(r) + L^2/(2\mu r^2)$.

d) [2.5 pt] Show that there exist a maximum angular momentum ℓ_{max} above which no bound orbits are possible. Give an expression for ℓ_{max} in terms of μ, k, a and numerical constants. Your answer must be explicit and must not contain exponential functions.

Note: I originally provided an incorrect solution, assuming that an orbit with $E > 0$ is equivalent to an unbound orbit. This is incorrect for the Yukawa potential, see e.g. the Fall 2002 Mechanics exam [2].

From Eq. (26), $\dot{r}^2 = \frac{2}{\mu}(E - V_{\text{eff}}(r))$. A bound orbit exists in minima of V_{eff} (see illustration below). Therefore, no bound orbits can exist if V_{eff} has no local minima. That is, $V'_{\text{eff}}(r) \neq 0$ everywhere.

$$V'_{\text{eff}}(r) = -\frac{L^2}{\mu r^3} + \frac{k}{r}e^{-r/a}\left(\frac{1}{r} + \frac{1}{a}\right) \tag{27}$$

This has no solution $V'_{\text{eff}}(r) = 0$ if:

$$\frac{L^2}{\mu r^3} > \frac{k}{r}e^{-r/a}\left(\frac{1}{r} + \frac{1}{a}\right) \text{ everywhere, i.e.} \tag{28}$$

$$L^2 > \mu ka\left[\frac{r}{a}\left(1 + \frac{r}{a}\right)e^{-r/a}\right] \text{ everywhere} \tag{29}$$

In the first line, we do not need to consider the other case $\frac{L^2}{\mu r^3} < -V'_{\text{eff}}(r)$ because $V'_{\text{eff}}(r)$ decays exponentially, while $\frac{L^2}{\mu r^3}$ decays as a power law, so it is impossible to have $\frac{L^2}{\mu r^3} < -V'_{\text{eff}}(r)$ everywhere.

To satisfy Eq. (29), $L^2/(\mu ka)$ has to be greater than the largest possible value of $\frac{r}{a}\left(1 + \frac{r}{a}\right)e^{-r/a}$. This quantity is maximized at $r/a = (1 + \sqrt{5})/2$:

$$0 = \frac{d}{dr}\left[\frac{r}{a}\left(1 + \frac{r}{a}\right)e^{-r/a}\right] \tag{30}$$

$$= e^{-r/a}\left(1 + \frac{r}{a} - \left(\frac{r}{a}\right)^2\right) \tag{31}$$

$$\Rightarrow \frac{r}{a} = \frac{1 + \sqrt{5}}{2}\ , \tag{32}$$

and this quantity has maximum value $(2+\sqrt{5})\exp\left[-(1+\sqrt{5})/2\right] \approx 4.2\times e^{-1.6} \approx 4.2\times 0.2 = 0.84$, where we have used the "potentially useful data" below. Therefore,

$$\ell_{\rm max} \approx \sqrt{0.84}\sqrt{\mu k a} \approx 0.92\sqrt{\mu k a} \tag{33}$$

e) [2 pt] Find a differential equation for $u(\phi)$, where $u \equiv 1/r$;

From Eq. (26),

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{L}{\mu r^2}\right)^2\left(\frac{dr}{d\phi}\right)^2 = \frac{2}{\mu}\left[E - \frac{L^2}{2\mu r^2} - V(r)\right] \tag{34}$$

$$\frac{L^2}{2\mu}\left(\frac{du}{d\phi}\right)^2 = \left[E - \frac{L^2}{2\mu}u^2 - V(u)\right] \tag{35}$$

where I have used $\frac{dr}{d\phi} = \frac{dr/dt}{d\phi/dt}$, $\frac{d\phi}{dt} = \frac{L}{\mu r^2}$, and $\frac{du}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi}$.

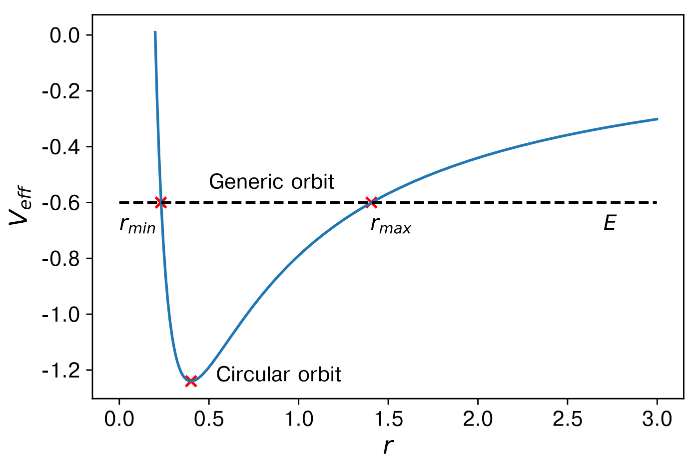
Consider now a situation where $\ell < \ell_{max}$ so that a stable circular orbit is possible at $r = r_0$. We will focus on an orbit which is very close to this one, i.e. $r(t) = r_0 + \delta r(t)$ with $|\delta r(t)| \ll r_0$.

f) [3 pt] Assuming $r_0 \ll a$, find an approximate expression for the advance of the perihelion in each revolution. Your solution must be of the form

$$\Delta\theta = \frac{A}{r_0^2} + \frac{B}{r_0} + Cr_0 + Dr_0^2 \tag{36}$$

Where A and B, C and D are real constants you must calculate (some or all of them might be zero) [Note: your answer cannot depend on k, μ or ℓ]

A crucial observation is the following: The intersection of the $V_{\rm eff}(r)$ curve with E describes the turning points $r_{\rm min}, r_{\rm max}$ of the orbit. Therefore, for a circular orbit, E intersects $V_{\rm eff}(r)$ at its minimum. (This is equivalent to the fact that a circular orbit minimizes E for a given L, and can also be seen from Eq. (26), demanding that $\dot{r} = 0$ and $\ddot{r} = 0$.)



After much thought, we realize that we are being asked to calculate the frequency (with respect to ϕ) of small oscillations about the minimum of $V_{\rm eff}$. Calling $\delta u = u - u_0 \approx -u_0^2\delta r$, the angle ϕ required for δu to make one oscillation is equal to $2\pi + \Delta\theta$, where $\Delta\theta$ is the advance of the perihelion.

Expanding Eq. (35) about u_0:

$$\frac{L^2}{2\mu}\left(\frac{d(\delta u)}{d\phi}\right)^2 \approx E - V_{\rm eff}(u_0) + \cancelto{0}{V'_{\rm eff}(u_0)}\,\delta u - \frac{1}{2}V''_{\rm eff}(u_0)\,(\delta u)^2 \tag{37}$$

$$= \delta E - \frac{1}{2}\left(2r_0^3\cancelto{0}{V'_{\rm eff}(r_0)} + r_0^4 V''_{\rm eff}(r_0)\right)(\delta u)^2 \tag{38}$$

Eq. (38) has solution $\delta u = A\cos(\Omega(\phi - \phi_0))$, with

$$\Omega^2 = \frac{\mu r_0^4}{L^2}V''_{\rm eff}(r_0) \tag{39}$$

To see this, differentiate Eq. (38) by ϕ and divide by $2(\delta u)'$ to obtain our friend $(\delta u)'' = -\Omega^2(\delta u)$.

Lastly we compute $V''_{\text{eff}}(r_0)$:

$$V_{\text{eff}}(r_0) = \frac{L^2}{2\mu r_0^2} - \frac{k}{r_0} e^{-r_0/a} = E \tag{40}$$

$$V'_{\text{eff}}(r_0) = -\frac{L^2}{\mu r_0^3} + \frac{k}{r_0} e^{-r_0/a} \left(\frac{1}{r_0} + \frac{1}{a} \right) = 0 \tag{41}$$

$$V''_{\text{eff}}(r) = 3\frac{L^2}{\mu r^4} - \frac{k}{r} e^{-r/a} \left[\left(\frac{1}{r} + \frac{1}{a} \right)^2 + \frac{1}{r^2} \right] \tag{42}$$

Using Eq. (41), Eq. (42) becomes:

$$V''_{\text{eff}}(r_0) = \frac{L^2}{\mu r_0^4} \left[3 - \left(1 + \frac{r_0}{a} + \frac{1}{1 + r_0/a} \right) \right] \tag{43}$$

$$\Omega^2 = 3 - \left(1 + \frac{r_0}{a} + \frac{1}{1 + r_0/a} \right) \tag{44}$$

$$\approx 1 - \left(\frac{r_0}{a} \right)^2 \tag{45}$$

Finally we obtain:

$$\Delta\theta = \frac{2\pi}{\Omega} - 2\pi \approx 2\pi \left(\frac{1}{\sqrt{1 - (r_0/a)^2}} - 1 \right) \approx \pi \left(\frac{r_0}{a} \right)^2 \tag{46}$$

So $A = B = C = 0$ and $D = \pi/a^2$. Treat yourself to a pie! To verify these answers, scroll down to the last line of Ref. [3].
— *Potentially useful data:*

$$\sqrt{2} = 1.4, \sqrt{3} = 1.7, \sqrt{5} = 2.2, \sqrt{7} = 2.6 \tag{47}$$

$$e^{-3.4} = 0.03, e^{-1.6} = 0.2, e^{-0.8} = 0.45, e^{-0.1} = 0.90 \tag{48}$$

This turns out to be important (but not sufficient?) for part d). What I initially thought was a red herring turned out to be a regular herring.
Note: The reasoning behind this question is likely to measure, through orbits, whether the gravitational potential is really a Yukawa potential (20) with a large a, by measuring the precession of a perihelion. By dimensional analysis and the hint, $\Delta\theta$ must be a function of r_0/a. Since a large a is undetectable, in hindsight A and B must be zero (not that it would have helped to solve this problem). I cannot think of a simple argument for why C should be 0.

References

[1] Cahit Erkal. The simple pendulum: a relativistic revisit. *European Journal of Physics*, 21(5):377–384, jul 2000. doi: 10.1088/0143-0807/21/5/302. URL https://doi.org/10.1088/0143-0807/21/5/302.

[2] URL http://physrefs.mit.edu/sites/default/files/documents/P2Fall02.pdf.

[3] URL http://www.physics.usu.edu/Wheeler/ClassicalMechanics/CMYukawaPotential.pdf.