Part II: Spring 2021 Electromagnetism Solutions

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January 29, 2023

1 Problem 1: Parallel Plates with an Indent

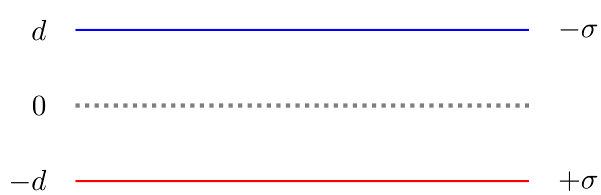
This problem is in SI units

Consider two parallel infinite conducting plates separated along the z direction by a distance d. The bottom plate is grounded and the upper plate is held at a potential such that there is a uniform electric field $E_0\hat{\mathbf{z}}$ between the plates. A hammer is struck against the bottom plate from below, leaving a hemispherical indent of radius $a \ll d$ (still held at zero potential) protruding towards the top plate.

(a) By considering suitable image charges, find the potential field between the plates $\Phi(r, \theta)$ in polar coordinates centered on the middle of the hemisphere. To find the potential, consider the following questions:

i) What arrangement of image charges would produce the uniform field between the plates without the indent present?

Adding a second infinite conducting plate at a distance $2d$ away from the first one, with an opposite charge density, will produce the desired effect:



Using Gauss's Law, the electric field between the plates, then, is $\mathbf{E}_{\text{in}} = (\sigma/\varepsilon_0)\hat{\mathbf{z}}$, which matches our given conditions. The potential this produces is proportional to z:

$$\Phi_{\text{in}}(z) = -\int_0^z \mathbf{E} \cdot \mathrm{d}\mathbf{z} = -\frac{\sigma}{\varepsilon_0}\mathrm{z}$$

ii) Now with the indent present, what image charges (that is, their magnitude and position) must be added to keep the indent at zero potential?

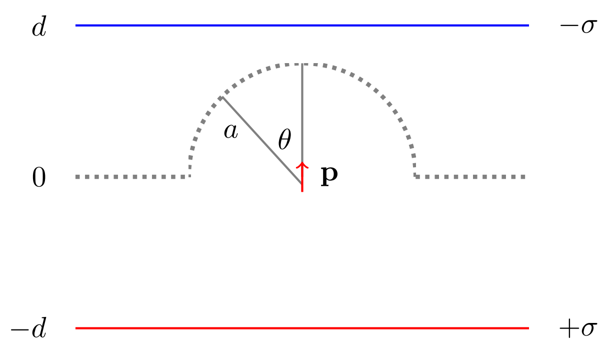
The indent adds a semicircle about the origin where the potential must stay grounded. As we saw from the previous part, the potential currently is proportional to z, so the new image charge must produce a stronger potential when $\theta = 0$, and no new potential when $\theta = \pi/2$. Clearly we need a function proportional to $\cos\theta$, which happens to be a perfect dipole. If the dipole is placed at the origin, then we have:

$$\Phi_{\text{dip}}(r, \theta) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

Which must cancel Φ_{in} when $r = a$. Using $z = a\cos\theta$:

$$\frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{a^2} = \frac{\sigma}{\varepsilon_0}a\cos\theta$$
$$\mathbf{p} = 4\pi\sigma a^3\hat{\mathbf{z}}$$

Our new configuration looks like (with the semicircle very exaggerated):



iii) What is the potential resulting from all of these charges?

We can simply add up Φ_{in} and Φ_{dip}, replacing z with $r\cos\theta$:

$$\Phi(r,\theta) = -\frac{\sigma}{\varepsilon_0}r\cos\theta + \frac{1}{4\pi\varepsilon_0}\frac{4\pi\sigma a^3\cos\theta}{r^2}$$

$$\boxed{\Phi(r,\theta) = \frac{\sigma}{\varepsilon_0}\left(\frac{a^3}{r^2} - r\right)\cos\theta}$$

iv) By enforcing that the field far from the indent, $a \ll r \ll d$, is not affected by the indent, and thus takes the original uniform form, determine the values of the image charges in terms of E_0 and geometric parameters.

We already found the electric field without the indent in part (i). By inspection, the charge density must be $\boxed{\sigma = \varepsilon_0 E_0}$. Plugging this into the dipole moment gives $\boxed{\text{p} = 4\pi\varepsilon_0 E_0 a^3\hat{\mathbf{z}}}$.

(b) Find the total charge induced on the hemispherical indent in the limit $a \ll d$.

The induced charge density is found simply by using the boundary condition:

$$\sigma_{\text{ind}} = -\varepsilon_0\frac{\partial\Phi}{\partial r}\bigg|_{r=a} = -\sigma\left(-2\frac{a^3}{r^3} - 1\right)\bigg|_{r=a}\cos\theta = 3\varepsilon_0 E_0\cos\theta$$

We can then find the total induced charge by integrating over the hemisphere

$$Q_{\text{ind}} = 3\varepsilon_0 E_0 a^2\int_0^{2\pi} d\phi\int_0^{\pi/2} d\theta\cos\theta\sin\theta = 3\varepsilon_0 E_0 a^2(2\pi)(1/2)$$

Therefore,

$$\boxed{Q_{\text{ind}} = 3\pi a^2\varepsilon_0 E_0}$$

2 Problem 2: Electromagnetic Angular Momentum

This problem is in Gaussian units

An infinitely long wire is on the z-axis and has a charge per unit length $-\lambda$. A plastic cylindrical shell of radius R is concentric about the z-axis and has a charge per unit area $\sigma = +\lambda/2\pi R$ uniformly distributed over its surface (and fixed to that surface). The cylindrical shell is suspended so that it can rotate freely about the z-axis without friction.

For $t < 0$, the shell is initially at rest and immersed in a constant magnetic field $\mathbf{B}_{\text{ext}} = B_0\hat{\mathbf{z}}$ produced by external currents on a concentric solenoid of radius $R_S \gg R$.

(a) What is the initial (i.e., $t < 0$) electric field for all $\rho < R_S$ (where $\boldsymbol{\rho}$ is the cylindrical radius vector)?

Using Gauss's Law, for $\rho < R$:

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} = 4\pi Q_{\text{enc}}$$
$$2\pi\rho\ell E = -4\pi\ell\lambda$$
$$E(\rho < R) = -\frac{2\lambda}{\rho}$$

Whereas for $R < \rho < R_S$, $Q_{\text{enc}} = -\lambda\ell + 2\pi R\ell\sigma = -\lambda\ell + \lambda\ell = 0$, so the electric field vanishes. In summary,

$$\boxed{\mathbf{E}(t<0) = \begin{cases} -2\lambda/\rho\,\hat{\boldsymbol{\rho}}\,, & \rho < R \\ 0\,, & R < \rho < R_S \end{cases}}$$

(b) The electromagnetic angular momentum density is given by

$$\boldsymbol{\mathcal{L}} = \frac{1}{4\pi c}\boldsymbol{\rho} \times (\mathbf{E} \times \mathbf{B}).$$

Calculate this and the find the total electromagnetic angular momentum per unit length along the z-axis for the initial configuration. Give your answer in terms of λ, B_0, R, and the speed of light, c.

First, let's calculate $\mathbf{E} \times \mathbf{B}$:

$$\mathbf{E} \times \mathbf{B} = -\frac{2\lambda}{\rho}B_0\hat{\boldsymbol{\rho}} \times \hat{\mathbf{z}} = \frac{2\lambda B_0}{\rho}\hat{\boldsymbol{\phi}} \quad (\text{for } \ \rho < R)$$

Now, with $\boldsymbol{\rho} = \rho\hat{\boldsymbol{\rho}}$:

$$\boldsymbol{\rho} \times (\mathbf{E} \times \mathbf{B}) = 2\lambda B_0\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\phi}} = 2\lambda B_0\hat{\mathbf{z}}$$

Remembering our different regions for $\mathbf{E}$,

$$\boxed{\boldsymbol{\mathcal{L}} = \begin{cases} (\lambda B_0/2\pi c)\hat{\mathbf{z}}\,, & \rho < R \\ 0\,, & R < \rho < R_S \end{cases}}$$

Now to get the total angular momentum per unit length, we integrate over the cross-sectional volume of the cylinder:

$$\begin{aligned}\mathbf{L} = \int \boldsymbol{\mathcal{L}}\,\mathrm{d}\tau &= \frac{\lambda B_0}{2\pi c}\int_0^R \mathrm{d}\rho\; 2\pi\rho\ell\hat{\mathbf{z}} \\ &= \frac{\lambda B_0 \ell}{c}\cdot\left.\frac{\rho^2}{2}\right|_0^R \hat{\mathbf{z}}\end{aligned}$$

Which gives a total angular momentum per unit length of:

$$\boxed{\frac{\mathbf{L}}{\ell} = \frac{\lambda B_0 R^2}{2c}\hat{\mathbf{z}}}$$

(c) Beginning at $t = 0$, the external current in the solenoid is slowly reduced to zero over some time $t_0 \gg R/c$, so that $\mathbf{B}_{\text{ext}} = \hat{\mathbf{z}}B_{\text{ext}}(t)$ with $B_{\text{ext}}(0) = B_0$ and $B_{\text{ext}}(t_0) = 0$. The shell will begin to rotate (assume that the shell is so massive that it rotates very slowly, and any self-magnetic field generated by its rotation may be neglected). Use the torque generated by the induced electric field to calculate the angular momentum per unit length of the shell at time $t = t_0$. How does this compare to the electromagnetic angular momentum at $t = 0$?

The induced electric field is given by Faraday's law, which reads (in integral form) as

$$\oint \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} = -\frac{1}{\mathrm{c}}\frac{\mathrm{d}}{\mathrm{dt}}\oint \mathbf{B}\cdot \mathrm{d}\mathbf{a}$$

Integrating around a circular loop of radius ρ,

$$2\pi\rho E_{\text{ind}} = -\frac{1}{c}\pi\rho^2\frac{\mathrm{d}B_{\text{ext}}}{\mathrm{d}t}$$

Using the right-hand rule, this field should be azimuthal (in the $\hat{\boldsymbol{\phi}}$ direction):

$$\mathbf{E}_{\text{ind}} = -\frac{\rho}{2c}\frac{\mathrm{d}B_{\text{ext}}}{\mathrm{d}t}\hat{\boldsymbol{\phi}}$$

The force per unit length generated by this electric field is $\mathbf{f} = \lambda\mathbf{E}_{\text{ind}}$, which in turn produces a torque per unit length:

$$\boldsymbol{\tau} = \lambda\boldsymbol{\rho}\times\mathbf{E}_{\text{ind}} = -\frac{\lambda R^2}{2c}\frac{\mathrm{d}B_{\text{ext}}}{\mathrm{d}t}\hat{\mathbf{z}}$$

Where I have plugged in $\rho = R$ since this is the radius where the shell is located. Of course, torque is also the time derivative of angular momentum, $\boldsymbol{\tau} = \mathrm{d}\mathbf{L}/\mathrm{dt}$, which we can exploit to find the angular momentum per unit length:

$$\frac{\mathbf{L}}{\ell} = -\frac{\lambda R^2}{2c}\hat{\mathbf{z}}\int_{t=0}^{t=t_0}\mathrm{dB}_{\text{ext}} = -\frac{\lambda \mathrm{R}^2}{2\mathrm{c}}(\mathrm{B}_{\text{ext}}(\mathrm{t}_0) - \mathrm{B}_{\text{ext}}(0))\hat{\mathbf{z}}$$

Plugging in the given values for $B_{\text{ext}}(0)$ and $B_{\text{ext}}(t_0)$, we find that the angular momentum per unit length matches our answer from part (b), confirming that it is conserved over time:

$$\boxed{\frac{\mathbf{L}}{\ell} = \frac{\lambda B_0 R^2}{2c}\hat{\mathbf{z}}}$$

(d) The flux density of electromagnetic angular momentum across a surface whose normal is $\hat{\boldsymbol{\rho}}$ is given by $\mathcal{F} = -(\boldsymbol{\rho} \times \overleftrightarrow{T}) \cdot \hat{\boldsymbol{\rho}}$, where $\overleftrightarrow{T}$ is the Maxwell stress tensor, given by

$$\overleftrightarrow{T} = \frac{1}{4\pi}\left[\mathbf{EE} + \mathbf{BB} - \frac{1}{2}\overleftrightarrow{I}(E^2 + B^2)\right],$$

where $\overleftrightarrow{I}$ is the identity matrix. Use this to calculate the total electromagnetic angular momentum per unit length that flows across an imaginary cylindrical shell at $\rho = R - \epsilon$ from $t = 0$ to $t = t_0$ ($\epsilon \ll 1$). How does this compare to your answer in (c)?

The cross product between a vector and a tensor will produce another tensor, which when dotted with $\hat{\boldsymbol{\rho}}$ will produce a vector. Given this information, we need only compute the cross product between $\boldsymbol{\rho}$ and the ρ-component of $\overleftrightarrow{T}$, i.e. $(T_{\rho\rho}, T_{\rho\phi}, T_{\rho z})$. Evaluating the necessary tensor components, given that E only has ρ and ϕ components and B only has a z component:

$$\begin{aligned} T_{\rho\rho} &= \frac{1}{8\pi}(E_\rho^2 - E_\phi^2 - B_z^2) \\ T_{\rho\phi} &= \frac{1}{4\pi}E_\rho E_\phi \\ T_{\rho z} &= 0 \end{aligned}$$

And this gives:

$$(\boldsymbol{\rho} \times \overleftrightarrow{T}) \cdot \hat{\boldsymbol{\rho}} = (\rho, 0, 0) \times (T_{\rho\rho}, T_{\rho\phi}, 0) = \hat{\mathbf{z}}\rho T_{\rho\phi}$$

Working out $T_{\rho\phi}$:

$$T_{\rho\phi} = \frac{1}{4\pi}\left(-\frac{2\lambda}{\rho}\right)\left(-\frac{\rho}{2c}\frac{\mathrm{d}B_{\text{ext}}}{\mathrm{d}t}\right) = \frac{\lambda}{4\pi c}\frac{\mathrm{d}B_{\text{ext}}}{\mathrm{d}t}$$

Thus,

$$\mathcal{F} = -\frac{\lambda\rho}{4\pi c}\frac{\mathrm{d}B_{\text{ext}}}{\mathrm{d}t}\hat{\mathbf{z}}$$

We now integrate over the area $\ell\rho\mathrm{d}\phi$ (*not* $\mathrm{d}\rho$ — $\mathcal{F}$ is the angular momentum flux density, meaning the angular momentum per *area* per time, so we just plug in $\rho = R - \epsilon$) and over $\mathrm{d}t$ to get the total electromagnetic angular momentum per unit length that flows across the cylinder at $R - \epsilon$:

$$\frac{\mathbf{L}(\rho)}{\ell} = -\frac{\lambda\rho^2}{4\pi c}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\mathrm{t}_0}\mathrm{dB}_{\text{ext}}\hat{\mathbf{z}}$$

Which gives:

$$\boxed{\frac{\mathbf{L}}{\ell} = \frac{\lambda B_0}{2c}(R-\epsilon)^2\hat{\mathbf{z}} \approx \frac{\lambda B_0}{2c}(R^2 - 2R\epsilon)\hat{\mathbf{z}}}$$

This matches the answer from part (c), which confirms that all of the electromagnetic angular momentum that was originally stored in the fields at $t = 0$ (from part b) has flowed outwards through the cylinder and been converted into mechanical angular momentum to make it rotate.