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DOCTORAL GENERAL EXAMINATION

PART II

FEBRUARY 5, 1998

FIVE HOURS

1. This examination is divided into four sections, Mechanics, Electricity & Magnetism, Statistical Mechanics, and Quantum Mechanics, with two problems in each. It is advisable to carefully read both problems in each section before making your choice. Submit ONLY one problem per section. IF YOU SUBMIT MORE THAN ONE PROBLEM FROM A SECTION, BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.

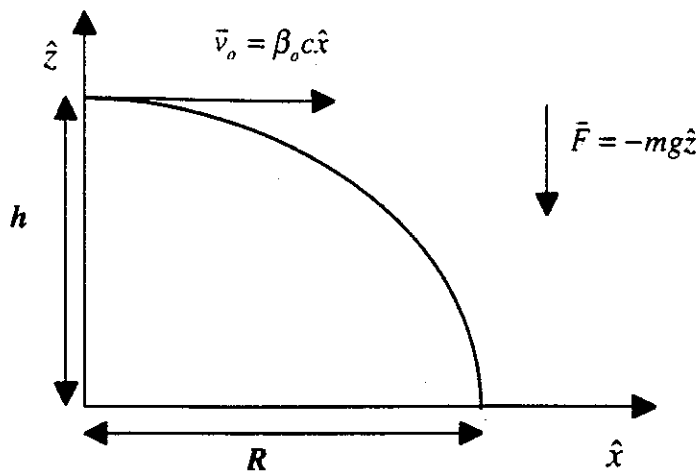
2. Use a separate fold of paper for each problem, and write your name on each fold. Include the problem number with each solution.

3. Calculators may be used.

4. **No Books or Reference Materials May Be Used.**

Mechanics I

A particle of mass m is launched at relativistic velocity $v_o = \beta_o c$ along the x axis from a height h as shown below. It then falls under the influence of Earth's gravity, landing a distance R away.



a) Give the relativistic Hamiltonian for the system and find Hamilton's equations of motion.

b) Find R in terms of h, m, β_o and fundamental constants by eliminating t in Hamilton's equations from part a. The integral

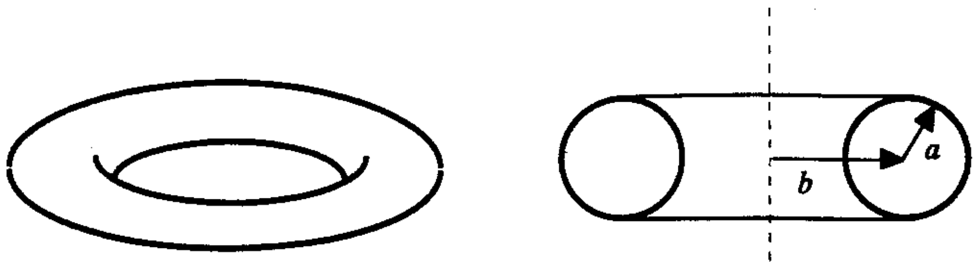
$$\int \frac{dx}{\sqrt{x^2 - p^2}} = \log\left(x + \sqrt{x^2 - p^2}\right)$$

will be useful.

c) Show your result from part b) gives the nonrelativistic range in the limit $\gamma_o = \dfrac{1}{\sqrt{1-\beta_o^2}} \to 1$ and $mc^2 >> mgh$.

Mechanics II

Consider a torus with dimensions as shown below ($b>a$). A particle is constrained to move on the surface of the torus. There are no applied forces.



a) Find a set of generalized coordinates to describe the particle's position on the torus. (Hint: be careful! There is one choice which is better than others!) Give the Lagrangian for the particle in terms of the generalized coordinates.

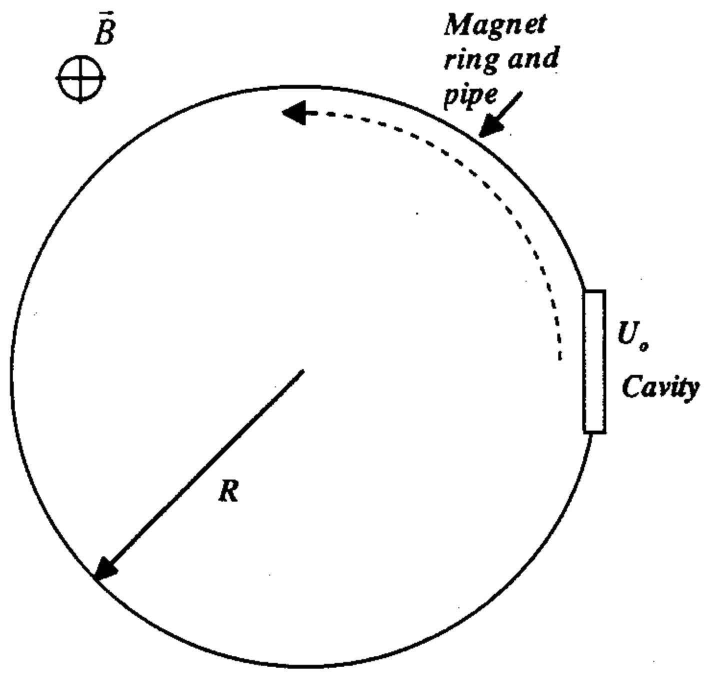
b) Find the conjugate momenta for the generalized coordinates from part *a*) and determine which generalized momenta are conserved.

c) Find the Hamiltonian for the particle on the torus and find the time rate of change for all the generalized coordinates. Find the time rate of change for their conjugate momenta.

d) Give an expression for each of the generalized momenta as a function of the generalized coordinates and other momenta. There are two distict types of motion which the particle may undergo. Draw representative trajectories of each on a sketch of the torus and on a phase diagram. Give a condition on the energy which separates the two types of motion.

EM I

A circular storage ring works in the following way: a particle starts at one end of a cavity with potential drop U_o. Once the particle crosses the cavity, the potential is switched off. The particle is bent in a perpendicular magnetic field around in a circle so it arrives back at the cavity, at which point the potential difference U_o is switched on.



This procedure is repeated, causing the particle to steadily gain energy. The magnetic field is increased each orbit to keep the orbit radius constant at R.

a) Find the magnetic field necessary to confine a particle of mass m_0 and charge q in terms of R, $\gamma = E/m_0c^2$, $\beta = p_c/E$ and m_o.

Assume the particle does not radiate due to acceleration.

b) Assume that at $t=0$ the particle is at rest at the end of the cavity. At $t=0$, the potential is switched on, accelerating the particle across the cavity. What is the energy of the particle after n orbits?

c) Treat n as a continuous variable and derive a differential equation for dE/dt. Find the energy of the particle as a function of time in terms of U_o, m, R, q, γ, β and fundamental constants.

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Now consider the radiation due to acceleration. The relativistic Larmor formula for the power radiated by a charge e in circular motion is

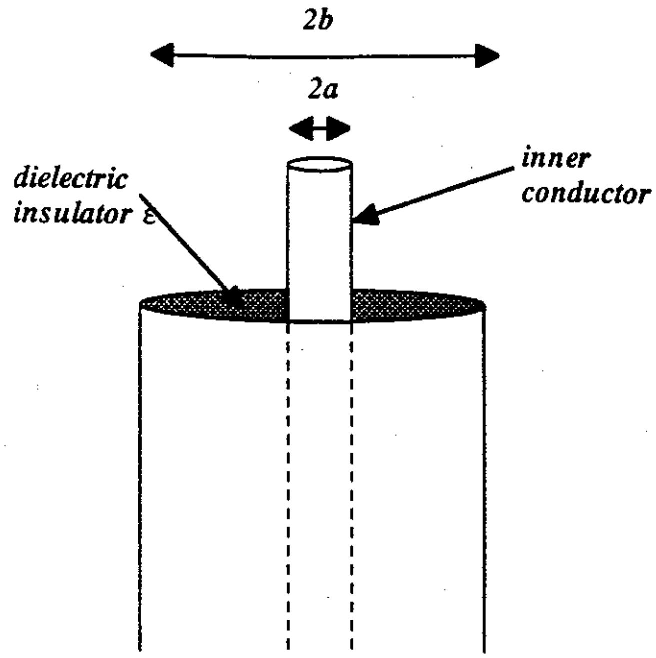
$$P = \frac{2}{3}\frac{e^2}{m^2 c^3}\gamma^2 \omega^2 |\bar{p}|^2$$

d) What is the total energy radiated by a particle in one revolution due to the acceleration in the magnetic field only (i.e. neglect radiation due to acceleration across the cavity) in terms of U_o, m, R, q, γ, β and fundamental constants?

e) For a given potential drop U_o, what is the maximum energy to which a particle may be accelerated? Assume the particle is ultrarelativistic, that is $\beta=1$ and express your answer in terms of U_o, m, R, q, γ, β and fundamental constants.

EM II

In the 1850's, the *Great Eastern* laid the first transAtlantic telegraph cable of approximately L=3,000 miles length. The cable consisted of a conductor surrounded by an insulator with dielectric constant ε as shown below. The conducting sea water surrounding the cable served as the second conductor, forming a coaxial waveguide.



Guided electromagnetic waves propagate inside the waveguide.

a) Indicate the direction of the electric and magnetic fields for a sinuisoidally varying propagating EM wave. Indicate the direction of the Poynting vector.

b) Find the inductance and capacitance per unit length of the cable. Assume the dielectric constant does not vary with frequency.

c) For a sinusoidally varying wave with frequency ω and wave number k propagating in the cable , write down expressions for the electric field $\vec{E}$ and magnetic field $\vec{B}$ in the cable. Use the fields to find the impedence of the cable, $Z_o = V/I$.

Assume the dielectric constant of the insulator varies as a function of frequency such that $\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$.

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d) Find the group and phase velocities as a function of ω. Explain the implications of your results in the limits $\omega \to \infty$ and $\omega \to \omega_p$. What range of frequencies may be transmitted over this cable? What happens when $\omega < \omega_p$?

Statistical Mechanics I

A large number of classical atoms of mass m are in equilibrium within a closed container at temperature T. The atoms can exist either as a gas occupying the volume V of the container or be adsorbed onto the inner surface area A of the container. Each adsorbed atom has a potential energy, $-\varepsilon_o$, because of its binding to the surface, but is otherwise free to move anywhere along (but not perpendicular to) the surface.

(a) The system is dilute enough that interatomic interactions can be ignored. Find the grand partition functions for the gas and surface systems.

(b) Derive the adsorbed atom surface density, n_s, in terms of pressure P temperature, atomic mass, ε_o, and fundamental constants. Sketch surface density n_s vs. P for high and low T.

(c) Suppose n_s can be large enough that it is affected by Lennard-Jones type ($V(r)=\left(-\frac{A}{r^6}+\frac{B}{r^{12}}\right)$) interaction between the adsorbed atoms. Assume the gas remains ideal. Discuss qualitatively and show how the interactions will affect the sketches in part (b) for low, intermediate and high values of P. What physical phenomenon would you expect to observe as you lower the temperature of the system?

(d) Can the surface system be non-ideal while the gas system is an ideal gas (as assumed in part c) and both are in equilibrium with each other? Explain.

Statistical Mechanics II

N non-interacting identical bosons of mass m are in equilibrium within a closed container at temperature T. Each particle can be either in a gas phase occupying the volume V of the container or be adsorbed on one of the N_a adsorption sites on the inside surface of the container. Each adsorption site can be either empty or be occupied by a single particle. The energy of an adsorbed particle is ω below the single particle ground-state energy in the gas. Take $\beta = 1/kT$

(a) Find grand partition function for the surface system and derive an expression for the total number of adsorbed particles in terms of N, ω and fugicity $f \equiv e^{\beta\mu}$.(Hint: the grand partition function for a single site is $1 + f\exp(\omega\beta)$.)

(b) Starting from the Bose-Einstein distribution, derive an expression for N_g/V, where N_g is the number of bosons in the gas, in terms of T, m, f, V and fundamental constants. Note that $\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{y^{1/2}dy}{e^{x+y}-1} = \sum_{n=1}^{\infty}\frac{e^{-nx}}{n^{3/2}}$.

(c) Let $G \equiv \sum_{n=1}^{\infty} 1/n^{3/2}$ be a positive finite constant. For what values of temperature T and nominal density N/V does Bose condensation occur? In the limit of small ω sketch your answer in the space of ($N/V,T$). Explain what happens if you cool down the gas when N/V is in the range $N_a/V \gtrsim N/V > N_a/2V$.

QM I

Ehrenfest's theorem says in the limit in which

$$\left\langle \frac{dV}{dx} \right\rangle = \frac{dV(\langle x \rangle)}{d\langle x \rangle} \qquad \text{(I)}$$

the expectation value $\langle x \rangle$ follows the classical time dependence.

Consider a quantum mechanical simple harmonic oscillator in one dimension. The SHO has mass m and spring constant k.

a) Give the hamiltonian. Show the potential of the quantum SHO fulfills condition (I)?

b) Give the normalized ground state and first excited state wave functions and their energies.

At t=0, the SHO is prepared with a wavefunction identical to the ground state but displaced by an amount b from the equilibrium position.

c) Use the translation operator $U(b)$ to construct an expression for the state of the SHO at t=0. Find the expectation value of the position $\langle x(t) \rangle$ and the momentum $\langle p(t) \rangle$ and show they follow the classical equations of motion.

d) Compute $\langle x(t)^2 \rangle$ and use this to express an approximate condition for the SHO to be in the classical regime.

QM II

Consider a proton-antiproton atom bound by Coulomb interactions.

a) Give the binding energy (in electronvolts), average relative velocity in the ground state (numerically expressed as a fraction of c) and the r.m.s. radius (in centimeters) of the ground state.

b) What are the possible values of the total angular momentum, j, and its z component, m_j, in the ground state and first excited state multiplets?

c) Suppose a very weak interaction, H' turns a proton into an antiproton (and *vice versa*) without affecting their spins:

$$H'|p_s\bar{p}_{\bar{s}}\rangle = \varepsilon|\bar{p}_s p_{\bar{s}}\rangle$$

with ε real and positive.
Draw and label a diagram showing the energy shifts of the ground state and excited state multiplets due to this interaction.

d) The states of the $p\bar{p}$ atom can decay into one another electromagnetically, by photon emission. Redraw the level diagram of part c) showing the allowed electric dipole and magnetic dipole transitions.

e) Now consider strong interaction effects. When the proton and antiproton are within a separation r_o they annihilate into mesons with unit probability. Estimate the lifetime of the ground state in terms of r_o, m and fundamental constants. Express your estimate of the width of the ground state as a fraction of its binding energy if r_o=10^{-13}cm.