General Exam Spring 1998 Solutions

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Mechanics Problem 1

(a)

For a relativistic particle, the Hamiltonian has the following form:

$$H = \sqrt{p_x^2c^2 + p_z^2c^2 + m^2c^4} + mgz = E \tag{1}$$

(b)

Write down the first half the Hamilton's equations:

$$\dot{p_x} = -\frac{\partial H}{\partial x} = 0 \tag{2}$$

$$\dot{p_z} = -\frac{\partial H}{\partial z} = -mg \tag{3}$$

$$\tag{4}$$

By solving the first order differential equations above:

$$p_x = p \tag{5}$$

$$p_z = -mgt + p_z^0 \tag{6}$$

Second half of the Hamilton's equations:

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{c^2 p_x}{\sqrt{p_x^2c^2 + p_z^2c^2 + m^2c^4}} \tag{7}$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{c^2 p_z}{\sqrt{p_x^2c^2 + p_z^2c^2 + m^2c^4}} \tag{8}$$

Given the initial conditions: $\dot{x}|_{t=0} = \beta_0 c$, $\dot{z}|_{t=0} = 0$, we can also determine that $p_z(t = 0) = 0$ thus $p_z^0 = 0$. Consider equation (6) at $t = 0$:

$$\beta_0 c = \frac{c^2 p}{\sqrt{p^2c^2 + m^2c^4}} \tag{9}$$

$$\frac{1}{\beta_0} = \frac{\sqrt{p^2c^2 + m^2c^4}}{pc} \tag{10}$$

$$\frac{1}{\beta_0^2} = 1 + \frac{m^2c^4}{p^2c^2} \tag{11}$$

$$\frac{m^2c^4}{p^2c^2} = -1 + \frac{1}{\beta_0^2} \tag{12}$$

$$\frac{p^2c^2}{m^2c^4} = \frac{1}{\frac{1}{\beta_0} - 1} = \frac{\beta_0^2}{1-\beta_0^2} = \beta_0^2\gamma_0^2 \tag{13}$$

Thus the expression of p in terms of β_0 is $p = \beta_0\gamma_0 mc$. Thus:

$$p^2c^2 + m^2c^4 = \beta_0^2\gamma_0^2 m^2c^4 + m^2c^4 \tag{14}$$

$$= m^2c^4\left(\frac{\beta_0^2}{1-\beta_0^2} + 1\right) \tag{15}$$

$$= m^2c^4\gamma_0^2 \tag{16}$$

The equation of motion becomes:

$$\dot{x} = \frac{\beta_0 \gamma_0 m c^3}{\sqrt{m^2 c^4 \gamma_0^2 + m^2 g^2 c^2 t^2}} \tag{17}$$

$$\dot{z} = \frac{-mgc^2 t}{\sqrt{m^2 c^4 \gamma_0^2 + m^2 g^2 c^2 t^2}} \tag{18}$$

simplify a little bit:

$$\dot{x} = \frac{\beta_0 \gamma_0 c^2}{g\sqrt{\frac{c^2\gamma_0^2}{g^2} + t^2}} \tag{19}$$

$$\dot{z} = \frac{-ct}{\sqrt{\frac{c^2\gamma_0^2}{g^2} + t^2}} \tag{20}$$

Solve the ODEs with initial positions $x|_{t=0} = 0$, $z|_{t=0} = h$:

$$x = \frac{\beta_0\gamma_0 c^2}{g} \int_0^t \frac{dt}{\sqrt{\frac{c^2\gamma_0^2}{g^2} + t^2}} \tag{21}$$

$$= \frac{\beta_0\gamma_0 c^2}{g} \ln[\, t + \sqrt{\frac{c^2\gamma_0^2}{g^2} + t^2}\Big|_0^t \,] \tag{22}$$

$$= \frac{\beta_0\gamma_0 c^2}{g} \ln\left[t + \sqrt{\frac{c^2\gamma_0^2}{g^2} + t^2}\right] - \ln\frac{c\gamma_0}{g} \tag{23}$$

$$z - h = -c\int_0^t \frac{tdt}{\sqrt{\frac{c^2\gamma_0^2}{g^2} + t^2}} \tag{24}$$

$$z = h - c\sqrt{\frac{c^2\gamma_0^2}{g^2} + t^2}\Big|_0^t \tag{25}$$

$$= h - c\sqrt{\frac{c^2\gamma_0^2}{g^2} + t^2} + \frac{c^2\gamma_0}{g} \tag{26}$$

Find the time t_0 when the particle hits the floor ($z = 0$):

$$c\sqrt{\frac{c^2\gamma_0^2}{g^2} + t_0^2} = \frac{c^2\gamma_0}{g} + h \tag{27}$$

$$\sqrt{\frac{c^2\gamma_0^2}{g^2} + t_0^2} = \frac{c\gamma_0}{g} + \frac{h}{c} \tag{28}$$

$$\frac{c^2\gamma_0^2}{g^2} + t_0^2 = \frac{c^2\gamma_0^2}{g^2} + \frac{2\gamma_0 h}{g} + \frac{h^2}{c^2} \tag{29}$$

Thus,

$$t_0 = \sqrt{\frac{2\gamma_0 h}{g} + \frac{h^2}{c^2}} \tag{30}$$

Plug t_0 into x and we get:

$$\boxed{R = \frac{\beta_0\gamma_0 c^2}{g}\ln\left[1 + \frac{hg}{c^2\gamma_0} + \sqrt{\frac{2gh}{\gamma_0 c^2} + \frac{h^2 g^2}{c^4\gamma_0^2}}\right]} \tag{31}$$

(c)

In the limit where $c \to \infty$;

$$R = \frac{vc}{g}\sqrt{\frac{2gh}{c^2}} = v\sqrt{\frac{2h}{g}} \tag{32}$$

The classical result is recovered. One can easily check by $\frac{1}{2}gt^2 = h$ and $R = vt$.

Mechanics Problem 2

(a)

One of the easiest solution to this problem is to define a typical toroidal coordinates. This set of coordinates is frequently used in fusion and plasma physics. The definition of angles may differ from textbooks to textbooks by different conventions. Here the toroidal coordinates are defined in the following figure:

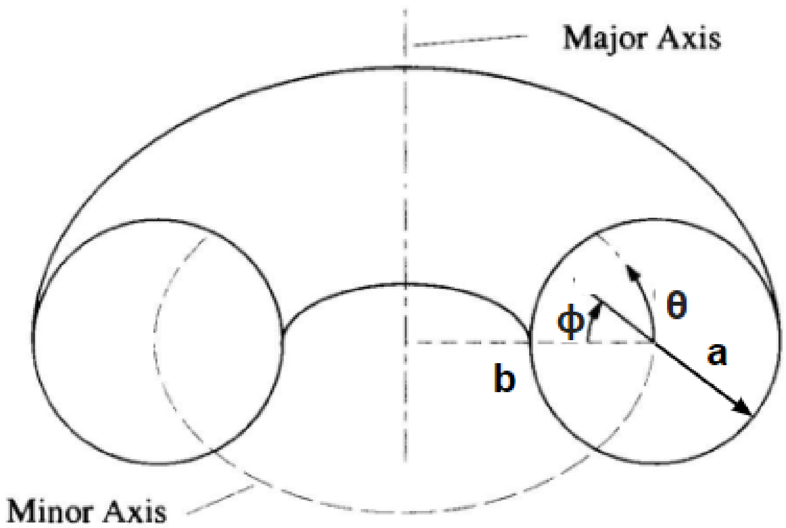

Figure 1: Definition of toroidal coordinates

Following the coordinates indicated above, we write can right down the kinetic energy:

$$T = \frac{1}{2}m(a\dot{\phi})^2 + \frac{1}{2}m(\rho\dot{\theta})^2 \tag{33}$$

where $\rho = b - a\cos\phi$.

$$T = \frac{1}{2}m(a\dot{\phi})^2 + \frac{1}{2}m(b - a\cos\phi)^2\dot{\theta}^2 \tag{34}$$

Since there's no potential, $\boxed{L = T = \frac{1}{2}m(a\dot{\phi})^2 + \frac{1}{2}m(b - a\cos\phi)^2\dot{\theta}^2}$.

(b)

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = m(b - a\cos\phi)^2\dot{\theta} = m\rho^2\dot{\theta} \tag{35}$$

$$p_\phi = ma^2\dot{\phi} \tag{36}$$

L is cyclic in θ, so p_θ is conserved.

(c)

$$\begin{aligned} H &= p_\theta\dot{\theta} + p_\phi\dot{\phi} - L && (37)\\ &= \frac{p_\theta}{2}\left(\frac{p_\theta}{m\rho^2}\right) + \frac{p_\phi}{L}\left(\frac{p_\phi}{ma^2}\right) && (38)\\ &= \frac{p_\theta^2}{2m\rho^2} - \frac{p_\phi^2}{2ma^2} && (39)\\ &= \frac{p_\theta^2}{2m(b - a\cos\phi)^2} - \frac{p_\phi^2}{2ma^2} && (40) \end{aligned}$$

$$\dot{p_\theta} = -\frac{\partial H}{\partial \theta} = 0 \tag{41}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m(b - a\cos\phi)^2} \tag{42}$$

$$\dot{p_\phi} = -\frac{\partial H}{\partial \phi} = \frac{p_\theta^2 a \sin\phi}{m(b - a\cos\phi)^3} \tag{43}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{ma^2} \tag{44}$$

(d)

$$\frac{dp_\phi/dt}{d\phi/dt} = \frac{dp_\phi}{d\phi} = \frac{p_\theta^2 a^3 \sin\phi}{(b - a\cos\phi)^3 p_\phi} \tag{45}$$

$$\int_{p_\phi(\phi_0)}^{p_\phi(\phi')} p_\phi dp_\phi = \int_{\phi_0}^{\phi'} \frac{p_\theta^2 a^3 \sin\phi}{(b - a\cos\phi)^3} d\phi \tag{46}$$

Do the integral on RHS by substitution: $u = b - a\cos\phi$. $du = a\sin\phi d\phi$.

$$\frac{1}{2}(p_\phi^2(\phi') - p_\phi^2(\phi_0)) = p_\theta^2 a^2 \int_{b-a\cos\phi_0}^{b-a\cos\phi'} \frac{du}{u^3} \tag{47}$$

$$= p_\theta^2 a^2 \left[\frac{1}{(b - a\cos\phi_0)^2} - \frac{1}{(b - a\cos\phi')^2}\right] \tag{48}$$

$$p_\phi(\phi') = p_\theta a \sqrt{\left[\frac{2}{(b - a\cos\phi_0)^2} - \frac{2}{(b - a\cos\phi')^2}\right] + p_\phi^2(\phi_0)} \tag{49}$$

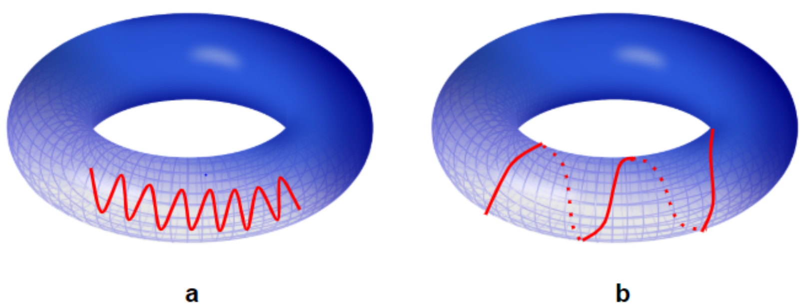

Figure 2: two types of motions in toroidal coordinates

In figure a, the particle oscillates around $\phi = \pi$ on the outer surface. If the energy of the particle is higher, the particle may start to wind around the torus as indicated in figure b. The separation is defined as $p_\phi(0) = 0$. Thus:

$$p_\theta a \sqrt{\left[\frac{2}{(b - a\cos\phi_0)^2} - \frac{2}{(b - a\cos\phi')^2}\right]} = p_\phi(\phi_0) \tag{50}$$

is the critical condition in terms of momentum. Rewriting it in terms of energy gives the desired answer in the question. However, it seems like from the original solution, full credit was given even if the answer is not expressed in terms of energy.

Electromagnetism Problem 1

(a)

$$\frac{d\vec{p}}{dt} = \frac{e}{c}\vec{v} \times \vec{B} \tag{51}$$

$$\vec{p} = m\gamma\vec{v} \tag{52}$$

$$\omega|\vec{p}| = \frac{e}{c}\frac{|p|B}{m\gamma} \tag{53}$$

$$\omega = \frac{eB}{\gamma m_0 c} = \frac{v}{R} \tag{54}$$

$$B = \frac{v\gamma m_0 c}{eR} \tag{55}$$

$$\boxed{B = \frac{\beta\gamma m_0 c^2}{eR}} \tag{56}$$

(b)

$$E = m_0 c^2 + nqU_0 \tag{57}$$

(c)

$$dn = \frac{vdt}{2\pi R} \tag{58}$$

$$\frac{dE}{dt} = qU_0\frac{dn}{dt} = \frac{qU_0}{2\pi R}v \tag{59}$$

$$E = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \frac{v^2}{c^2} = 1 - \frac{m_0^2 c^4}{E^2} \tag{60}$$

$$v = \frac{c}{E}\sqrt{E^2 - m_0^2 c^4} \tag{61}$$

$$\frac{dE}{dt} = \frac{qU_0 c}{2\pi R}\frac{\sqrt{E^2 - m_0^2 c^4}}{E} \tag{62}$$

Solve the differential equation:

$$\frac{EdE}{\sqrt{E^2 - m_0^2 c^4}} = \frac{qU_0 c}{2\pi R}t \tag{63}$$

$$\int_{E(0)}^{E(t)} \frac{EdE}{\sqrt{E^2 - m_0^2 c^4}} = \frac{qU_0 c}{2\pi R}t \tag{64}$$

$$\sqrt{E^2 - m_0^2 c^4}\Big|_{E(0)=m_0 c^2}^{E(t)} = \frac{qU_0 c}{2\pi R}t \tag{65}$$

$$\sqrt{E^2(t) - m_0^2 c^4} = \frac{qU_0 c}{2\pi R}t \tag{66}$$

$$\boxed{E(t) = \sqrt{m_0^2 c^4 + (\frac{qU_0 c}{2\pi R}t)^2}} \tag{67}$$

(d)

$$T = \frac{2\pi R}{v} \tag{68}$$

$$E = PT = \frac{2\pi R}{v}\frac{2}{3}\frac{q^2}{m^2c^3}\gamma^2\omega^2|\vec{p}|^2 \tag{69}$$

$$= \frac{4\pi}{3}\frac{R}{v}\frac{q^2}{m^2c^3}(\frac{v}{R})^2\gamma^2 m^2\gamma^2 v^2 \tag{70}$$

$$= \frac{4\pi}{3}\frac{q^2}{c^3}\frac{v^3}{R}\gamma^4 \tag{71}$$

$$= \boxed{\frac{4\pi}{3}\frac{q^2}{R}\beta^3\gamma^4} \tag{72}$$

(e)

$$qU_0 = \frac{4\pi}{3}(\frac{q^2}{R})\beta^3\gamma^4 \tag{73}$$

$$\tag{74}$$

as $\beta \to 1$, $U_0 = \frac{4\pi}{3}\frac{q}{R}\gamma^4$. Thus,

$$E = m_0c^2\gamma = \boxed{m_0c^2[\frac{3U_0}{4\pi}\frac{R}{q}]^{\frac{1}{4}}} \tag{75}$$

Electromagnetism Problem 2

(a)

The TEM mode electric and magnetic fields of a coaxial cable is indicated in the following figure:

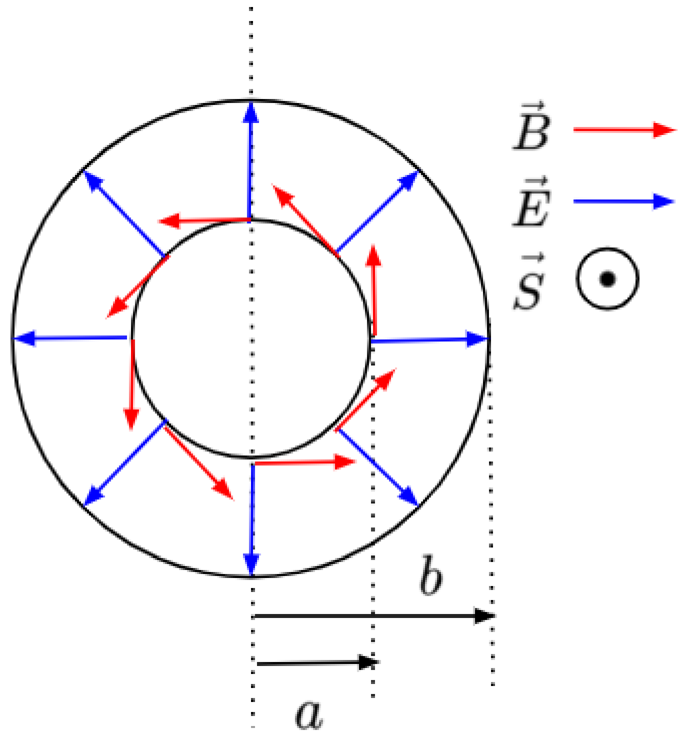

Figure 3: TEM mode of a coaxial cable

The electric field $\vec{E}$ is in radial direction pointing outwards and the magnetic field $\vec{B}$ is in tangential direction pointing counterclockwise. The Poynting vector $\vec{S}$ points out of the plane.

(b)

Capacitance:

$$\epsilon \oiint \vec{E} \cdot d\vec{a} = 4\pi Q \tag{76}$$

$$\epsilon E(r) 2\pi r = 4\pi\lambda \tag{77}$$

$$E(r) = \frac{2\lambda}{\epsilon}\frac{1}{r} \tag{78}$$

$$V = \frac{2\lambda}{\epsilon}\ln\frac{b}{a} \tag{79}$$

$$Q_0 = C_0 V \tag{80}$$

$$\boxed{C_0 = \frac{\epsilon}{2}\frac{1}{\ln\left(\frac{b}{a}\right)}} \tag{81}$$

Inductance:

$$\mathcal{E} = -L\frac{dI}{dt} = -\frac{1}{c}\frac{d\Phi_B}{dt} \tag{82}$$

Let I be the current. Recall that $\vec{\nabla} \times \vec{B} = \frac{4\pi j}{c}$.

$$2\pi r B = \frac{4\pi I}{c} \tag{83}$$

$$B = \frac{2I}{cr} \tag{84}$$

$$\Phi_0 = \int_a^b B(r)dr = \frac{2I}{c} \ln \frac{b}{a} \tag{85}$$

$$\boxed{L_0 = \frac{d\Phi_0}{cdI} = \frac{2}{c^2} \ln \frac{b}{a}} \tag{86}$$

(c)

We need to equations from the Maxwell's equations:

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{87}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{\epsilon}{c}\frac{\partial \vec{E}}{\partial t} \tag{88}$$

For TEM mode in wave guides, the electric field and magnetic field have the form of electrostatic solution in two-dimension multiplied by a plane wave. The electrostatics are as indicated in section (a):

$$\vec{E} = \frac{2\lambda}{\epsilon}\frac{1}{r} \sin(kx - \omega t)\hat{\rho} \tag{89}$$

$$\vec{B} = \frac{2I}{c}\frac{1}{r} \sin(kx - \omega t)\hat{\phi} \tag{90}$$

From the Maxwell's equations above:

$$k|E| = \frac{\omega}{c}|B| \tag{91}$$

$$k|B| = \frac{\epsilon\omega}{c}|E| \tag{92}$$

$$\Rightarrow k^2 = \epsilon(\frac{\omega}{c})^2 \tag{93}$$

$$\boxed{k = \sqrt{\epsilon}\frac{\omega}{c}} \tag{94}$$

$$\boxed{\frac{|E|}{|B|} = \frac{1}{\sqrt{\epsilon}}} \tag{95}$$

$$\Rightarrow \frac{2\lambda/\epsilon}{2I/c} = \frac{1}{\sqrt{\epsilon}} = \frac{\lambda c}{I\epsilon} \tag{96}$$

$$\boxed{\frac{\lambda}{I}c = \sqrt{\epsilon}} \tag{97}$$

$$V = \frac{2\lambda}{\epsilon} \ln \frac{b}{a} \tag{98}$$

$$\frac{V}{I} = \frac{2}{\epsilon}\frac{\lambda}{I} \ln \frac{b}{a} = \boxed{\frac{2}{c} \ln \frac{b}{a} \frac{1}{\sqrt{\epsilon}} = Z} \tag{99}$$

(d)

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \tag{100}$$

$$\tag{101}$$

Phase velocity:

$$v_{ph} = \frac{\omega}{k} = \frac{c}{\sqrt{\epsilon}} = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} \tag{102}$$

group velocity:

$$v_g = \frac{d\omega}{dk} \tag{103}$$

$$k^2 = (1 - \frac{\omega_p^2}{\omega^2})\frac{\omega^2}{c^2} \tag{104}$$

$$c^2k^2 = \omega^2 - \omega_p^2 \tag{105}$$

$$2c^2k = 2\omega\frac{d\omega}{dk} \tag{106}$$

$$\frac{d\omega}{dk} = c^2\frac{k}{\omega} = c^2\frac{\sqrt{\epsilon}}{c} = c\sqrt{\epsilon} \tag{107}$$

$$v_g = c\sqrt{\epsilon} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}} \tag{108}$$

The behaviours of phase velocity (blue) and group velocity (orange) are shown in the diagram. The electromagnetic wave is transmitted when $\omega > \omega_p$. When $\omega < \omega_p$ the dielectric constant is negative, which basically indicates that the electric field and the displacement points in opposite directions. This enables interesting applications such as metamaterials.

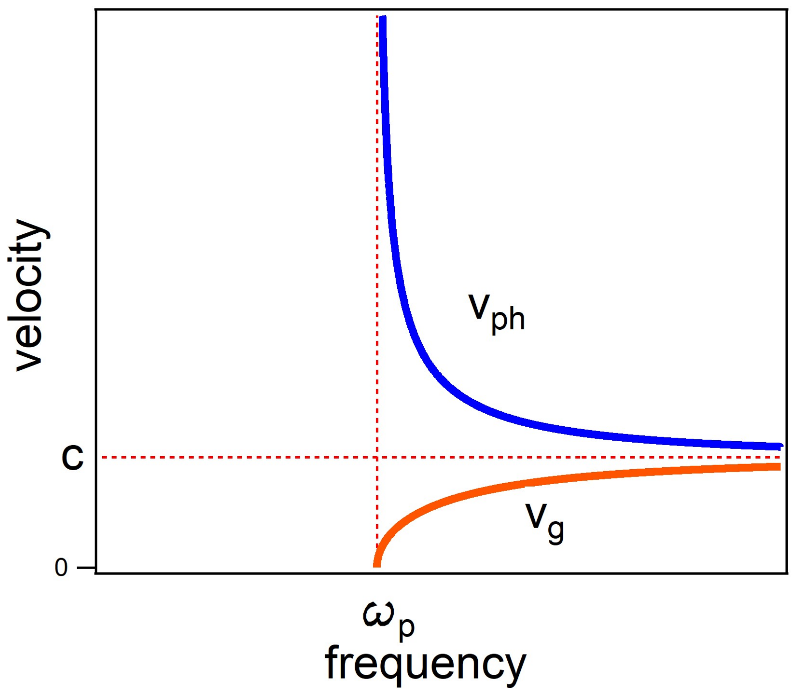

Figure 4: group velocity and phase velocity as a function of frequency.

Statistical Mechanics Problem 1

(a)

For the gas system:

$$\mathcal{Z}_g(\beta, V, N_g) = \sum_{N_g=0}^{\infty} e^{\beta\mu N_g} Z_g(\beta, V, N_g) \tag{109}$$

Where $Z_g(\beta, V, N_g)$ is the canonical partition function:

$$Z_g(\beta, V, N_g) = \frac{1}{N_g!}\frac{1}{h^{3N_g}}\left[\int d^3r d^3p e^{-\beta\frac{p^2}{2m}}\right]^{N_g} = \frac{1}{N_g!}\left[\frac{V}{\lambda_T^3}\right]^{N_g} \tag{110}$$

where $\lambda_T^2 = \frac{h^2\beta}{2\pi m}$ is the thermal wavelength. Thus,

$$\mathcal{Z}_g(\beta, V, N_g) = \sum_{N_g=0}^{\infty} \frac{1}{N_g!}\left[\frac{Ve^{\beta\mu}}{\lambda_T^3}\right]^{N_g} = \boxed{\exp\left(\frac{V}{\lambda_T^3}e^{\beta\mu}\right)} \tag{111}$$

For the surface system:

$$\mathcal{Z}_s(\beta, A, N_s) = \sum_{N_g=0}^{\infty} e^{\beta\mu N_s} Z_s(\beta, A, N_s) \tag{112}$$

The canonical partition function is:

$$Z_s(\beta, A, N_s) = \frac{1}{N_s}!\frac{1}{h^{2N_s}}\left[\int d^2r d^2p e^{-\beta\frac{p^2}{2m}} e^{\beta\varepsilon_0}\right]^{N_s} = \frac{1}{N_s!}\left[\frac{A}{\lambda_T^2}e^{\beta\varepsilon_0}\right]^{N_s} \tag{113}$$

Thus,

$$\mathcal{Z}_s(\beta, A, N_s) = \sum_{N_s=0}^{\infty} \frac{1}{N_s!}\left[\frac{Ae^{\beta\mu}e^{\beta\varepsilon_0}}{\lambda_T^2}\right]^{N_s} = \boxed{\exp\left(\frac{A}{\lambda_T^2}e^{\beta\mu}e^{\beta\varepsilon_0}\right)} \tag{114}$$

(b)

$$n_s = \frac{N_s}{A} = \frac{1}{A}\frac{\partial}{\partial(\beta\mu)}\ln \mathcal{Z}_s(\beta, A, N_s) = \frac{1}{\lambda_T^2}e^{\beta\mu}e^{\beta\varepsilon_0} \tag{115}$$

When in equilibrium, the surface and the gas should have same chemical potential and temperature, thus $e^{\beta\mu}$ can be eliminated. Since

$$\beta PV = \ln \mathcal{Z}_g(\beta, V, N_g) = \frac{V}{\lambda_T^3}e^{\beta\mu} \tag{116}$$

Then

$$\boxed{n_s = P\frac{h}{k_B T(2\pi m k_B T)^{\frac{1}{2}}}e^{\frac{\varepsilon_0}{k_B T}}} \tag{117}$$

(c)

At low P the density is low and there should be no effects; at intermediate P, the attractive nature of the interaction will yield an effective larger ε_0 so that $n_s(P)$ will have a steeper slope; at high P, $n_s(P)$ will saturate due to the short range repulsive nature of the interaction. As the temperature lowers, a phase transition is expected.

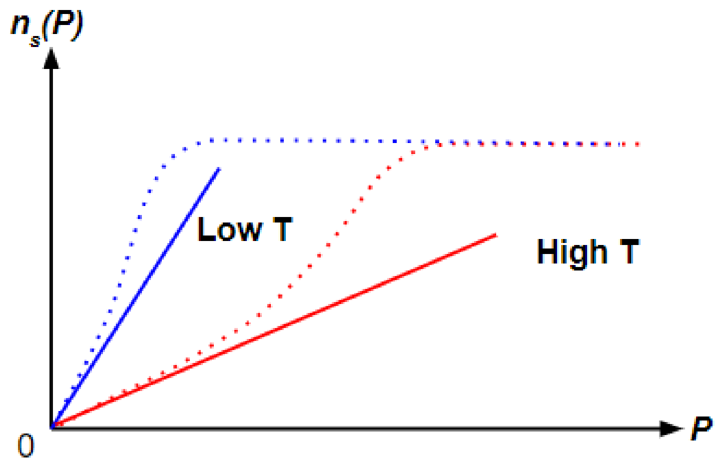

Figure 5: Skectch for (b) (solid lines) and (c) (dashed lines)

(d)

Yes, when $\varepsilon_0 \gg k_B T$, the inter-atomic separation on the surface is shorter so that it makes a non-ideal system.

Statistical Mechanics Problem 2

(a)

The grand partition function for the surface system is just

$$\boxed{\mathcal{Z}_s = (1 + f e^{\beta\omega})^{N_a}} \tag{118}$$

and the average number of absorbed site, N_A is then

$$N_A = f\frac{\partial \ln \mathcal{Z}_s}{\partial f} = N_A\frac{f e^{\beta\omega}}{1 + f e^{\beta\omega}} = \boxed{\frac{N_a}{1 + e{-}\beta(\omega+\mu)} \text{ or } \frac{N_a}{1 + f^{-1}e^{-\beta\omega}}} \tag{119}$$

(b)

$$N_g = \int_0^\infty \frac{d\varepsilon D(\varepsilon)}{f^{-1}e^{\beta\varepsilon} - 1} + \frac{1}{f^{-1} - 1} \tag{120}$$

The density of stats of a free particle $D(\varepsilon)$ can be obtained from $D(\varepsilon)d\varepsilon = \frac{4\pi k^2 dk}{\frac{(2\pi)^3}{V}} = \frac{V}{2\pi^2}k^2 dk$. Using $\varepsilon = \frac{\hbar^2 k^2}{2m}$ and $\frac{dk}{d\varepsilon} = \frac{m}{\hbar^2 k}$, $\Rightarrow D(\varepsilon) = \frac{2}{\sqrt{\pi}}V(\frac{2\pi m}{h^2})^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}$. Thus,

$$\frac{N_g}{V} = \frac{2}{\sqrt{\pi}}(\frac{2\pi m}{h^2})^{\frac{3}{2}}\int_0^\infty \frac{d\varepsilon\varepsilon^{\frac{1}{2}}}{f^{-1}e^{\beta\varepsilon} - 1}\frac{\beta^{\frac{3}{2}}}{\beta^{\frac{3}{2}}} + \frac{V^{-1}}{f^{-1} - 1} \tag{121}$$

$$= \frac{2}{\sqrt{\pi}}\frac{1}{\lambda_T^3}\int_0^\infty \frac{y^{\frac{1}{2}}dy}{f^{-1}e^{y} - 1} + \frac{V^{-1}}{f^{-1} - 1} \tag{122}$$

where $y = \varepsilon\beta$ and $\lambda_T^3 \equiv (\frac{h^2\beta}{2\pi m})^{\frac{3}{2}}$. Using the identity that

$$\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{y^{\frac{1}{2}}dy}{f^{-1}e^{y} - 1} = \sum_{n=1}^{\infty}\frac{f^n}{n^{\frac{3}{2}}} \tag{123}$$

we obtain

$$\boxed{\frac{N_g}{V} = \frac{1}{\lambda_T^3}\sum_{n=1}^{\infty}\frac{f^n}{n^{\frac{3}{2}}} + \frac{V^{-1}}{f^{-1} - 1}} \tag{124}$$

(c)

Combining (a) and (b) gives $\frac{N_g}{V} = \frac{N - N_A}{V} = \frac{N}{V} - \frac{N_a/V}{a + f^{-1}e^{-\beta\omega}}$. Bose Condensation occurs when $f \to 1$ and the term $\frac{1}{V}\frac{1}{f^{-1}-1}$becomes significant. That is, when

$$\frac{N}{V} - \frac{N_a/V}{1 + e^{-\beta\omega}} > \frac{1}{\lambda_T^3}\sum_{n=1}^{\infty} n^{-\frac{3}{2}} = \frac{G}{\lambda_T^3} \tag{125}$$

Where G is some numerical constant. Thus,

$$\boxed{\frac{N}{V} > \frac{N_a/V}{1 + e^{-\beta\omega}} + \frac{G}{\lambda_T^3}} \tag{126}$$

In the limit that ω is small we have:

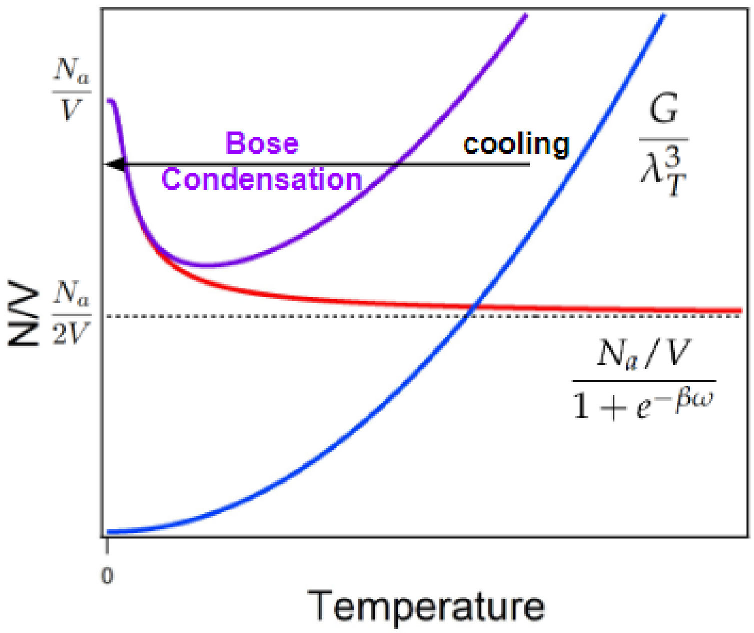

Figure 6: Region above the magenta curve is where Bose condensation occurs.

The magenta curve represents the phase boundary for Bose condensation. The region above it is where Bose condensation occurs. In the regime indicated in the question, when we cool down, the system first condensates and then de-condensates because of the depletion due to surface absorption.

Quantum Mechanics Problem 1

(a)

Hamiltonian for Simple Harmonic Oscillator:

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{127}$$

$$\frac{dV}{dx} = m\omega^2 x \tag{128}$$

$$\left\langle \frac{dV}{dx} \right\rangle = m\omega^2 \langle x \rangle \tag{129}$$

Compare:

$$V(\langle x \rangle) = \frac{1}{2}m\omega^2 \langle x \rangle^2 \tag{130}$$

$$\frac{dV(\langle x \rangle)}{d\langle x \rangle} = m\omega^2 \langle x \rangle \tag{131}$$

Thus we confirm the potential of SHO fulfills condition (I).

(b)

$$H = \frac{1}{2}\hbar\omega(aa^\dagger + a^\dagger a) \tag{132}$$

The creation and annihilation operators are defined such that:

$$\frac{1}{\sqrt{2}}(a + a^\dagger) = \frac{x}{l} \tag{133}$$

$$\frac{1}{\sqrt{2}}(a - a^\dagger) = \frac{ipl}{\hbar} \tag{134}$$

where l is a constant to be determined. Since $[x, p] = i\hbar$, $\frac{1}{2}2[a^\dagger, a] = \frac{i}{\hbar}[x, p] = -1$ follows as desired. Write x and p in terms of creation and annihilation operators:

$$x = \frac{l}{\sqrt{2}}(a + a^\dagger) \tag{135}$$

$$p = -\frac{i\hbar}{\sqrt{2}l}(a - a^\dagger) \tag{136}$$

substitute into the original Hamiltonian:

$$H = -\frac{\hbar^2}{2l^2}(-aa^\dagger - a^\dagger a)\frac{1}{2m} + \frac{1}{2}m\omega^2\frac{l^2}{2}(aa^\dagger + a^\dagger a) + a^2 \text{ and } a^{\dagger 2} \text{ terms} \tag{137}$$

square terms cancel if $-\frac{\hbar^2}{4ml^2} + \frac{1}{2}m\omega^2\frac{l^2}{2} = 0$. By solving this equation, we obtain $\boxed{l = \sqrt{\frac{\hbar}{m\omega}}}$. Then, the Hamiltonian reads:

$$\boxed{H = \hbar\omega(a^\dagger a + \frac{1}{2})} \tag{138}$$

And the energy levels of SHO simply follows $\boxed{E_n = (\frac{1}{2} + n)\hbar\omega}$

- Ground state:

we know that $a|0\rangle = 0$. In order to solve for the wave equation, we need to write it in terms of x and p in real space:

$$x + \frac{ipl^2}{\hbar} = \frac{l}{\sqrt{2}}a + \frac{l}{\sqrt{2}}a \tag{139}$$

$$x + \frac{ip}{m\omega} = \sqrt{2}la \tag{140}$$

$$a = \frac{1}{\sqrt{2}l}(x + \frac{i}{m\omega}p) \tag{141}$$

Apply a to the ground-state $|0\rangle$ and sandwich with $\langle x|$:

$$(x + \frac{\hbar}{m\omega}\frac{d}{dx})\psi_0(x) = 0 \tag{142}$$

$$\psi_0(x) = Ce^{-\frac{x^2}{2l^2}} \tag{143}$$

Normalize the wave function:

$$C^2 \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{l^2}} = C^2 l\sqrt{\pi} \tag{144}$$

$$C = \frac{1}{(\pi l^2)^{\frac{1}{4}}} = (\frac{m\omega}{\pi\hbar})^{\frac{1}{4}} \tag{145}$$

Thus the ground state wave function is:

$$\boxed{\psi_0(x) = (\frac{m\omega}{\pi\hbar})^{\frac{1}{4}} e^{-\frac{x^2}{2l^2}}} \text{ where } l = \sqrt{\frac{\hbar}{m\omega}} \tag{146}$$

- First excited state:

(Note: the original solution started by assuming the from of $\psi_1(x)$ to be $C_1 x e^{\frac{x^2}{2l^2}}$ and normalize the wave function directly from the ansatz. However, I find it not very informative. So, here I'll include another approach.)

The original solution provided by the committee:
We know $\psi_1(x) = C_1 x e^{\frac{x^2}{2l^2}}$, Find C_1:

$$\int_{-\infty}^{\infty} dx C_1^2 x^2 e^{-\frac{x^2}{l^2}} = 1 \tag{147}$$

$$C_1^2 l^3 \int_{-\infty}^{\infty} dy y^2 e^{-y^2} = 1 \tag{148}$$

$$\int_{-\infty}^{\infty} dy y^2 e^{-y^2} = -\frac{d}{d\alpha}\int_{-\infty}^{\infty} dy e^{-\alpha y^2}\bigg|_{\alpha=1} = -\frac{d}{d\alpha}\frac{1}{\sqrt{\alpha}}\sqrt{\pi}\bigg|_{\alpha=1} = \frac{\sqrt{\pi}}{2} \tag{149}$$

So $C_1^2 l^3 = \frac{2}{\sqrt{\pi}}$, $C_1 = \frac{1}{l}\frac{\sqrt{2}}{\pi^{1/4}}(\frac{m\omega}{\hbar})^{\frac{1}{4}}$

$$\boxed{\psi_1(x) = \sqrt{2}(\frac{m\omega}{\pi\hbar})^{1/4}\frac{x}{l}e^{-\frac{x^2}{2l^2}}} \tag{150}$$

Alternative solution provided by PhysREFS:
Start from operating creation operator on the ground state: $a^\dagger|0\rangle = |1\rangle$ Again, write $a^\dagger$ in terms of x and p and act in real space:

$$x - \frac{ipl^2}{\hbar} = \frac{l}{\sqrt{2}}a^\dagger + \frac{l}{\sqrt{2}}a^\dagger \tag{151}$$

$$a^\dagger = \frac{1}{\sqrt{2}l}(x - \frac{ipl^2}{\hbar}) = \frac{1}{\sqrt{2}l}(x - l^2\frac{d}{dx}) \tag{152}$$

$$\tag{153}$$

$$\psi_1(x) = a^\dagger \psi_0(x) \tag{154}$$
$$= \frac{1}{\sqrt{2}l}(x - l^2 \frac{d}{dx})(\frac{m\omega}{\pi\hbar})^{\frac{1}{4}} e^{-\frac{x^2}{2l^2}} \tag{155}$$
$$= \frac{1}{\sqrt{2}l}((\frac{m\omega}{\pi\hbar})^{\frac{1}{4}} x e^{-\frac{x^2}{2l^2}} - l^2(-\frac{2x}{2l^2})(\frac{m\omega}{\pi\hbar})^{\frac{1}{4}} e^{-\frac{x^2}{2l^2}}) \tag{156}$$
$$= \frac{\sqrt{2}}{l}(\frac{m\omega}{\pi\hbar})^{\frac{1}{4}} x e^{-\frac{x^2}{2l^2}} \tag{157}$$
$$= \boxed{\sqrt{2}(\frac{m\omega}{\pi\hbar})^{1/4} \frac{x}{l} e^{-\frac{x^2}{2l^2}}} \tag{158}$$

Which is (of course) consistent with the result above. Hope this alternative solution helps :)

(c)

$U(b) = e^{\frac{ipb}{\hbar}}$ translates the state by b. Heisenberg picture state $|b\rangle = U(b)|0\rangle$ is equivalent to Schrodinger picture at $t = 0$. We seek $\langle b| x(t) |b\rangle$ which is the expectation value of Heisenberg operator in Heisenberg state. Use quantum equations of motion for operators:

$$i\hbar\dot{x} = [x, H] = [x, \frac{p^2}{2m}] = i\hbar\frac{p}{m} \tag{159}$$
$$\dot{x} = \frac{p}{m} \tag{160}$$
$$i\hbar\dot{p} = [p, H] = [p, \frac{1}{2}m\omega^2 x^2] = -m\omega^2 x i\hbar \tag{161}$$
$$\dot{p} = -m\omega^2 x \tag{162}$$

We find that the equations of motion of quantum SHO are consistent with those of classical SHO. This can be easily checked by applying Hamilton's equations to the Hamiltonian.
Consider expectation values in Heisenberg picture:

$$\frac{d}{dt}\langle b| x(t) |b\rangle = \frac{1}{m}\langle b| p(t) |b\rangle \tag{163}$$
$$\frac{d}{dt}\langle b| p(t) |b\rangle = -m\omega^2 \langle b| p(t) |b\rangle \tag{164}$$

Define

$$\langle b| p(t) |b\rangle \equiv p_b(t) \tag{165}$$
$$\langle b| x(t) |b\rangle \equiv x_b(t) \tag{166}$$

Rewrite the equations above in the new definition:

$$\dot{x_b} = \frac{p_b}{m} \tag{167}$$
$$\dot{p_b} = -m\omega^2 x_b \tag{168}$$

Solve the system of differential equations:

$$x_b(t) = x_b(0)\cos\omega t + \frac{p_b(0)}{m\omega}\sin\omega t \tag{169}$$
$$p_b(t) = p_b(0)\cos\omega t - m\omega x_b(0)\sin\omega t \tag{170}$$

Now

$$\langle b| x(0) |b\rangle = \langle 0| U^\dagger(b) x U(b) |0\rangle \tag{171}$$
$$= \langle 0| x + b |0\rangle = b \tag{172}$$
$$\langle b| p(0) |b\rangle = \langle 0| U^\dagger(b) p U(b) |0\rangle \tag{173}$$
$$= \langle 0| p |0\rangle = 0 \tag{174}$$

Thus,

$$\boxed{x_b(t) = b\cos\omega t} \tag{175}$$
$$\boxed{p_b(t) = -m\omega b\sin\omega t} \tag{176}$$

(d)

Need $\langle b| x(t)^2 |b\rangle$. From the previous part:

$$x(t) = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t \tag{177}$$
$$p(t) = p(0)\cos\omega t - m\omega x(0)\sin\omega t \tag{178}$$
$$x(0) = x = \text{Schrodinger picture operator} \tag{179}$$
$$p(0) = p = \text{Schrodinger picture operator} \tag{180}$$
$$x(t)^2 = x^2\cos^2\omega t + \frac{p^2}{m^2\omega^2}\sin^2\omega t + \frac{\sin\omega t\cos\omega t}{m\omega}(xp+px) \tag{181}$$
$$\langle b| x^2 |b\rangle = \langle 0| (x+b)^2 |0\rangle = \langle x^2\rangle_0 + b^2 \tag{182}$$
$$\langle b| p^2 |b\rangle = \langle 0| p^2 |0\rangle = \langle p^2\rangle_0 \tag{183}$$

Note that odd terms in x or p are eliminated by symmetry.

$$\langle b| xp+px |b\rangle = \langle 0| (x+b)p + p(x+b) |0\rangle = \langle xp+px\rangle_0 \tag{184}$$
$$\tag{185}$$

But

$$\langle xp+px\rangle_0 = \int dx[\psi_0(x)(-i\hbar\psi_0'(x)) + (i\hbar\psi_0'(x))\psi_0(x)] = 0 \tag{186}$$

So

$$\langle b| x^2(t) |b\rangle = \langle x^2\rangle_0\cos^2\omega t + \frac{\langle p^2\rangle_0}{m^2\omega^2}\sin^2\omega t + b^2\cos^2\omega t \tag{187}$$

Using Viral theorem:

$$\langle\frac{1}{2}m\omega^2x^2\rangle_0 = \frac{1}{2m}\langle p^2\rangle_0 \tag{188}$$

we obtain:

$$\langle b| x^2(t) |b\rangle = b^2\cos^2\omega t + \langle x^2\rangle_0 \tag{189}$$

And the ground state energy:

$$\frac{1}{2}m\omega^2\langle x_0^2\rangle = \frac{1}{4}\hbar\omega \tag{190}$$
$$\langle x_0^2\rangle = \frac{\hbar}{2m\omega} \tag{191}$$

So,

$$\boxed{\langle b| x^2(t) |b\rangle = b^2\cos^2\omega t + \frac{\hbar}{2m\omega} = b^2\cos^2\omega t + \frac{1}{2}l^2} \tag{192}$$

Motion is classical when $b \gg l$, where l is the natural length scale of quantum oscillator.

Quantum Mechanics Problem 2

(a)

Hydrogen with $m_e \to \frac{m_p}{2}$, $\frac{m_p/2}{m_e} \cong 1000$. The binding energy:

$$E_B = \frac{1}{2}\alpha^2 mc^2 = 1000(13eV) = 13keV \tag{193}$$

Radius of ground states:

$$r = \frac{1}{1000} \times r_{\text{Hydrogen}} = 5 \times 10^{-9} cm/1000 = 5 \times 10^{-12} cm \tag{194}$$

From Virial theorem:

$$\left\langle \frac{p^2}{2m} \right\rangle = \frac{1}{2} \left\langle \frac{e^2}{r} \right\rangle \tag{195}$$

$$-E_B = \left\langle \frac{p^2}{2m} \right\rangle - \left\langle \frac{e^2}{r} \right\rangle = -\frac{1}{2} \left\langle \frac{e^2}{r} \right\rangle = -\left\langle \frac{p^2}{2m} \right\rangle \tag{196}$$

So $\left\langle \frac{p^2}{2m} \right\rangle = \frac{1}{2} mc^2\alpha^2$.
the average velocity:

$$\left\langle \frac{v^2}{c^2} \right\rangle = \alpha^2 \tag{197}$$

$$\left\langle \frac{v}{c} \right\rangle = \alpha =\cong \frac{1}{137} \tag{198}$$

(b)

Both proton and antiproton has spin $\frac{1}{2}$, the total spin is $S = 0, 1 \bigotimes L = 0, 1, \cdots$
Ground state:$N = 1, L = 0, S = 0, 1$. So

$$j = 0 \ (m_j = 0) \tag{199}$$

or

$$j = 1 \ (m_j = -1, 0, 1) \tag{200}$$

First excited states: $N = 2, L = 0, 1, S = 0, 1$. So

$$L = 0 : j = 0(m_j = 0) \tag{201}$$
$$j = 1(m_j = -1, 0, 1) \tag{202}$$
$$L = 1, S = 1 : j = 0(m_j = 0) \tag{203}$$
$$j = 1(m_j = -1, 0, 1) \tag{204}$$
$$j = 2(m_j = -2, -1, 0, 1, 2) \tag{205}$$
$$L = 1, S = 0 : j = 1(m_j = -1, 0, 1) \tag{206}$$
$$\tag{207}$$

(c)

An interaction that changes $p \leftrightarrow \bar{p}$ is equivalent to sending $\vec{r} \to -\vec{r}$ on the spatial wave function. Since the spin labels are unchanged,

$$H' \left| p_s \bar{p}_{\bar{s}} \right\rangle = \varepsilon \left| \bar{p}_s p_{\bar{s}} \right\rangle \tag{208}$$

The spins of proton and antiproton are exchanged. Now $\vec{r} \to -\vec{r}$ gives $(-1)^l$onY_m^l and $s_1 \leftrightarrow s_2$ gives $(-1)^{s+1}$ on spin states. For example: $s = 0$ state $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$ is odd under spin excahnge. $s = 1$ state $|\uparrow\uparrow\rangle$ is even. So,

$$\boxed{H' \left| LS \right\rangle = \varepsilon'(-1)^{L+S+1} \left| LS \right\rangle} \tag{209}$$

Thus,

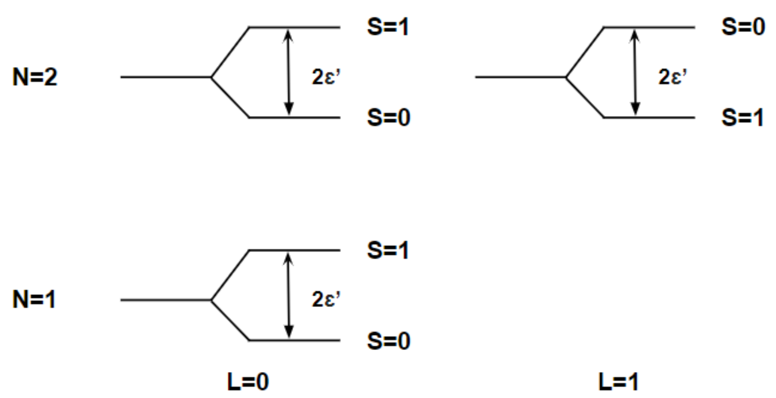

Figure 7: Energy levels of the bound states, labels with quantum numbers N, L and S

(d)

The electric dipole operator is $\vec{P} = e\vec{r}$, which has the following selection rules:

- Changes parity
- Couples like operator with $L = 1$ so $\Delta L = \pm 1$ ($\Delta L = 0$ not allowed because it doesn't change parity)
- Couples like operator with $J = 1$, so $\Delta J = 0, \pm 1$
- $\Delta S = 0$

Thus, the allowed transition are indicated as following:

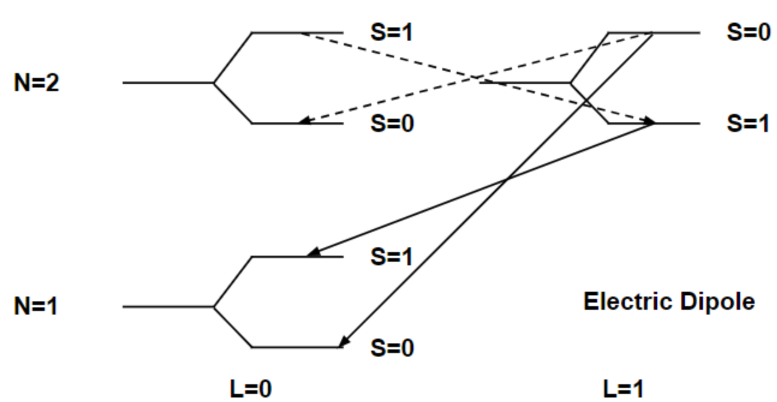

Figure 8: Allowed transitions for electric dipole

The solid lines indicate strong transitions ($\Delta N = 1$) while the dashed lines indicate weak transitions ($\Delta N = 0$).

Magnetic dipole operator is $\vec{M} \propto \vec{\sigma}$ and thus has the following selection rules:

- No parity change
- $\Delta L = 0$
- $\Delta J = 0, 1$
- $\Delta S = 1$

Thus, the allowed transition are indicated as following:

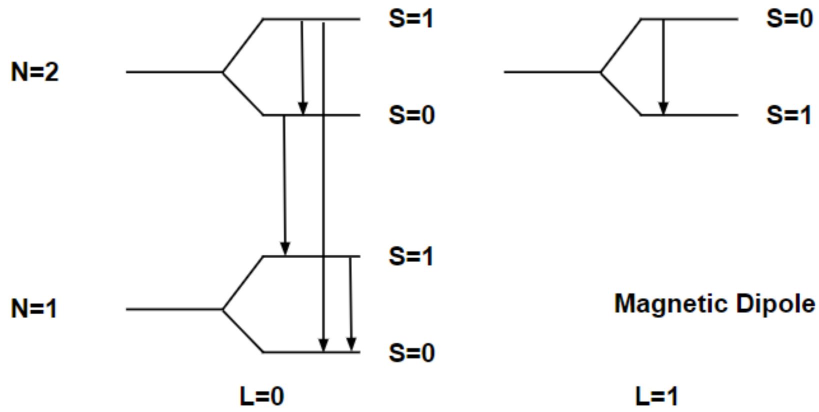

Figure 9: Allowed transitions for magnetic dipole

(e)

Probability to be inside r_0:

$$P(r_0) = \int d^3r|\psi(r)|^2\Theta(r_0 - r) \cong (\frac{r_0}{a_0})^3 \tag{210}$$

Time scale for the motion:

$$\frac{1}{T} = \frac{E_B}{\hbar} = \frac{\alpha^2 mc^2}{2\hbar} \tag{211}$$

Or

$$T \sim \frac{a_0}{r} = \frac{\hbar}{me^3}\frac{1}{\alpha c} = \frac{2\hbar}{\alpha^2 mc^2} \tag{212}$$

So lifetime $\frac{1}{\tau} \cong P(r_0)T^{-1}$

$$\tau \cong \frac{2\hbar}{\alpha^2 mc^2}(\frac{a_0}{r_0})^3 \tag{213}$$

Numerically, $\frac{r_0}{a_0} \cong \frac{1}{50}$, $\frac{\hbar}{mc} = 2 \times 10^{-14} cm$ (proton Compton wavelength). $\tau \cong (50)^3 \times 2 \times 10^{-14} \frac{2\times(137^2)}{3\times 10^{10} cm/s} \cong \frac{4}{3} \times 125 \times 10^3 \times 10^{-14} \times \frac{2\times 10^4}{10^{10}} \cong 3 \times 10^{-15} s$ Thus the width of ground state as a fraction of binding energy is

$$\frac{\Gamma}{E_B} = \frac{\hbar}{3 \times 10^{-15} \cdot 13 keV \cdot s} = 2 \times 10^{-5} \tag{214}$$