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DOCTORAL GENERAL EXAMINATION

PART II

February 5, 1999

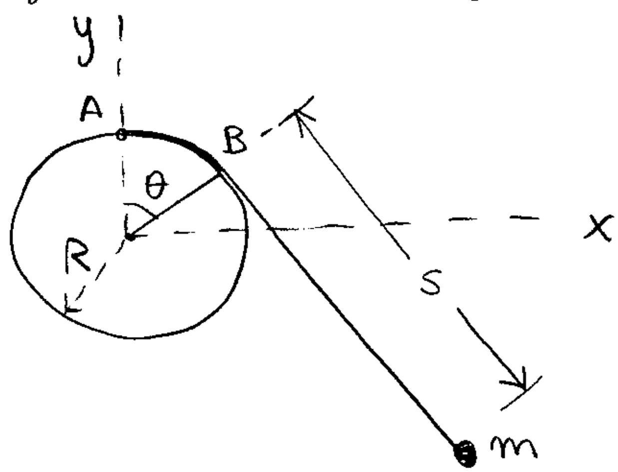
FIVE HOURS

1. This examination is divided into four sections, Mechanics, Electricity & Magnetism, Statistical Mechanics, and Quantum Mechanics, with two problems in each. It is advisable to carefully read both problems in each section before making your choice. Submit ONLY one problem per section. IF YOU SUBMIT MORE THAN ONE PROBLEM FROM A SECTION, BOTH WILL BE GRADED, AND THE PROBLEM WITH THE LOWER SCORE WILL BE COUNTED.
2. Use a separate fold of paper for each problem, and write your name on each fold. Include the problem number with each solution.
3. Calculators may be used.
4. **No Books or Reference Materials May Be Used.**

Problem 1 (Mechanics)

Consider a mass m attached to a string which in turn is nailed to point A on a circular spool of radius R. The whole system lies on the horizontal plane, and the spool is fixed so it cannot rotate. As the mass slides without friction the string remains taut and either winds or unwinds around the spool. B is the point where the string leaves the spool.

Let the total length of the string be l, and let s denote the free length of the string, that is, the length from B to the mass. We align the coordinate axes so that the center O of the spool corresponds to $x = y = 0$ and the radius to OA is in the positive y-direction. Let θ denote the angle between OA and OB.



(a) Express the coordinates (x, y) of the mass in terms of s, θ and R. Using the constraint relating s and θ to the total length, find the Lagrangian $L(s, \dot{s})$ of the system in terms of the dynamical coordinate s and its associated velocity $\dot{s}$.

(b) Find the Hamiltonian $H(s, p)$, write Hamilton's equations, and confirm that p/s is a constant of the motion. Use this information to find $s(t)$ in terms of its initial value s_0 and the total energy E.

(c) Consider the following change of coordinates:

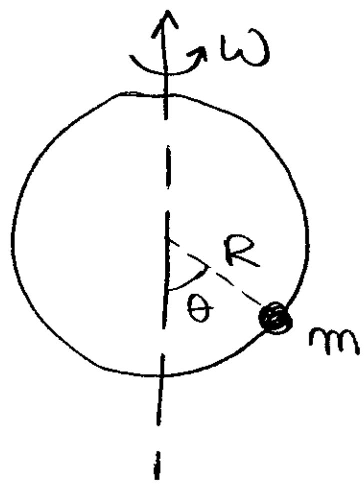
$$Q = s^2 , \qquad P = \frac{p}{\lambda s} ,$$

Find the value of the constant λ so that the transformation of variables is canonical. Give $H'(Q, P)$, and write the resulting Hamilton-Jacobi (HJ) equation for the action $S(Q, t)$.

(d) Assume a solution of the HJ equation of the form $S(Q, t) = W(Q) - \alpha t$. Solve for $W(Q)$ and use the fact that $\frac{\partial S}{\partial \alpha} = \beta =$ constant, to find $Q(t)$ (and thus $s(t)$).

Problem 2 (Mechanics)

A particle of mass m is constrained to move on a circular wire of radius R. The particle can slide without friction. The circular wire spins with constant angular velocity $\vec{\omega}$ about a vertical diameter. The particle feels the force of gravity (mg).



(a) Write the Lagrangian for the system using θ as the generalized coordinate. Identify an effective potential $V_{eff}(\theta)$ for the generalized coordinate θ.

(b) Write down the Euler Lagrange equations (do not solve them).

(c) Find constant values θ_i, $i = 1, 2, \cdots$, for which $\theta(t) = \theta_i$, is a solution of the equations of motion. Express your answer in terms of ω, R and g. Does the existence of any of these values depend on the magnitude of ω ?

(d) Consider now small oscillations around each of the θ_i identified in part (c). The oscillations may or may not be stable, and this may also depend on the magnitude of ω. Discuss the stability of each case, find the oscillation frequencies Ω_i and indicate critical values of ω if any.

Problem 1 (Electromagnetism)

A sphere of radius R has a total charge Q uniformly distributed on the surface. The sphere is rotated with angular velocity ω. Align the z axis along the angular velocity.

(a) Find the magnetic field throughout space. Do it by verifying that the following ansatz works: the magnetic field inside is a constant, and outside it is that of a pure magnetic dipole:

$$\vec{B} = B_0\,\vec{e}_z\,, \qquad r \leq R$$

$$\vec{B} = -\nabla\left(\frac{m\cos\theta}{r^2}\right) = 2m\,\frac{\cos\theta}{r^3}\,\vec{e}_r + m\,\frac{\sin\theta}{r^3}\,\vec{e}_\theta\,, \qquad r > R\,.$$

Use the continuity equations for the magnetic field across the surface of the sphere to find the values of B_0 and m in terms of ω, R, Q and the velocity of light c.

(b) Now assume ω is increased from zero with constant and small $\dot{\omega}$. The magnetic field will change in time. Use Faraday's law to calculate the induced electric field on $r = R$ as a function of θ. This electric field is in the $\vec{e}_\phi$ direction.

(c) The above induced electric field produces a torque $\vec{G}$ on the sphere. Calculate its magnitude and give its direction.

(d) Suppose now the sphere has a moment of inertia I_0 about the z-axis and is driven by an external mechanical torque $\vec{G}_{ext}$. Write down the equation of motion that determines $\frac{d\omega}{dt}$. The sphere will behave as if it had an additional moment of inertia I_{mag}. Calculate I_{mag} in terms of Q, R and the speed c of light.

Problem 2 (Electromagnetism)

We are given an alternating current source: $I(t) = I_0 \cos \omega t$, and a perfectly conducting wire of length d. We want to compare the radiation performance of the two antennas constructed as follows. In case (I) the wire is cut into two equal pieces and made into a center-fed linear antenna. In case (II) the wire is bent into a circular loop of perimeter d. We assume $\frac{\omega d}{c} = kd << 1$.

First analyze case (I). Align the z-axis along the antenna with the origin at P (the gap at P is assumed to be negligibly small). Assume the current distribution along the wire is given by

$$I(z,t) = I_0 \left(1 - \frac{\alpha_0 |z|}{d}\right) \cos \omega t \quad .$$

where α_0 is a constant

(a) Use charge conservation to find the the value of α_0 and to calculate the linear charge density $\lambda(z,t)$ on this antenna. Define the time-independent (possibly complex) $\lambda(z)$ via the usual relation $\lambda(z,t) = \Re\left\{\lambda(z)e^{-i\omega t}\right\}$. Give $\lambda(z)$ both for z positive and negative.

(b) Find the time-independent $\vec{p}$ for this center-fed radiating system (recall $\vec{p}(t) = \Re\left\{\vec{p}e^{-i\omega t}\right\}$. Give the total radiated power $P_{center-fed}$ in terms of I_0, k and d.

(c) Does this linear antenna radiate in the magnetic dipole term? Does it radiate in the electric quadrupole term? Explain your answers (set the origin at P).

Now analyze the circular antenna (II). Assume the current is independent of the position in the circle and is given by $I_0 \cos \omega t$.

(d) This antenna will radiate as a magnetic dipole. Find the total radiated power P_{circular} in this mode in terms of I_0, k and d. Compute the ratio $P_{\text{circle}}/P_{\text{center-fed}}$ in terms of kd. Is your answer reasonable?

(e) Does the circular antenna radiate in the electric dipole or electric quadrupole modes? Explain.

Useful Information:

* Total power radiated in the electric dipole mode: $\frac{ck^4}{3}|\vec{p}|^2$
* Total power radiated in the magnetic dipole mode: $\frac{ck^4}{3}|\vec{m}|^2$
* Magnetic moment of a current loop: $\frac{I(\text{Area})}{c}$.

Statistical Mechanics *Problem 1*

(a) An idealization of a vibrating violin string is a dynamical system of infinitely many harmonic oscillators, with frequencies $\omega_0, 2\omega_0, 3\omega_0, \ldots$, so that the energy levels of the system, after subtracting the zero point energies of the oscillators, are

$$E(n_1, n_2, n_3, \ldots) = \hbar\omega_0(n_1 + 2n_2 + 3n_3 + \cdots)$$

where each n_i takes values $0, 1, 2, \ldots$. That is, the energies are $E = \hbar\omega_0 N$, $N = 0, 1, 2, \ldots$ with degeneracy $p(N)$, the number of *partitions* of N. Obtain a physicist's quick estimate of the asymptotic behavior of $p(N)$ as $N \to \infty$ by calculating the free energy of the system at high temperature ($kT \gg \hbar\omega_0$) using the canonical ensemble and hence deriving the entropy-energy relation.

In fact the Hardy-Ramanujan formula for $p(N)$ is $p(N) \sim \frac{1}{4\sqrt{3}N} \exp \pi\sqrt{2N/3}$ as $N \to \infty$. Is your result consistent with this? $\left[\sum_1^\infty 1/n^2 = \pi^2/6\right]$

(b) In the theory of the relativistic massless string one finds a system with energy levels

$$E(n_1, n_2, n_3, \ldots) = \hbar\omega_0[n_1 + 2n_2 + 3n_3 + \cdots]^{1/2} .$$

Use the result of (a) to write down the entropy as a function of energy for $E \gg \hbar\omega_0$ and derive the temperature-energy relation. Describe what happens if this system is used as a heat bath, sharing energy with another system.

$$\left[\text{Thermodynamic relations:} \qquad S = -\frac{dF}{dT} \qquad E = F + TS\right]$$

Statistical Mechanics *Problem 2*

Consider a simplified model of the electronic structure of a semiconductor in which N electrons are distributed over $2N$ states, consisting of a zero-width valence band of N degenerate states and a zero-width conduction band of N degenerate states. The energy gap is Δ. The electrons are treated as noninteracting.

(a) Calculate the number of electrons in the conduction band as a function of temperature. What is the chemical potential?

(b) The electrons can make transitions between bands by emitting or absorbing photons. Show that the result of (a) and the black-body distribution of photons are consistent with the up and down transition rates being in balance.

(c) Now suppose the conduction band consists of N states spread uniformly in energy from Δ to 2Δ above the zero-width valence band. Calculate the chemical potential and the number of electrons in the conduction band at temperatures $\ll \Delta$.

Quantum Mechanics *Problem 1*

A system consists of three spin-$\frac{1}{2}$ particles. The spin angular momentum vector of particle r is $\mathbf{S}^{(r)} = \frac{1}{2}\hbar\boldsymbol{\sigma}^{(r)}$. The total spin is

$$\mathbf{S}^{\text{tot}} = \mathbf{S}^{(1)} + \mathbf{S}^{(2)} + \mathbf{S}^{(3)} \ .$$

The states $|m_1, m_2, m_3\rangle$ are eigenstates of $S_z^{(r)}$ with

$$S_z^{(r)} |m_1, m_2, m_3\rangle = m_r\hbar \, |m_1, m_2, m_3\rangle \ .$$

Consider the state $|\psi\rangle = \frac{1}{\sqrt{2}} \left|\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\rangle - \frac{1}{\sqrt{2}} \left|-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\rangle$.

(a) What is $\mathbf{S}^{\text{tot}} \cdot \mathbf{S}^{\text{tot}} |\psi\rangle$? [No calculation necessary.]

(b) Let $R = \left(\mathbf{S}_x^{\text{tot}} + i\mathbf{S}_y^{\text{tot}}\right)^3$. Explain why only one of the matrix elements $\langle m_1', m_2', m_3'| R |m_1, m_2, m_3\rangle$ is not zero. Show that $|\psi\rangle$ is an eigenstate of $R + R^\dagger$. What are the other eigenstates?

(c) Let

$$\begin{aligned} A &= \sigma_x^{(1)}\sigma_y^{(2)}\sigma_y^{(3)} \\ B &= \sigma_y^{(1)}\sigma_x^{(2)}\sigma_y^{(3)} \\ C &= \sigma_y^{(1)}\sigma_y^{(2)}\sigma_x^{(3)} \\ D &= \sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)} \ . \end{aligned}$$

Show that A, B, C, D all commute with each other and that $ABCD = \lambda 1$. What is λ?

(d) Show that $|\psi\rangle$ is an eigenstate of the operators A, B, C, D. What are the eigenvalues?

(e) The system is prepared in the state $|\psi\rangle$. An observer measures (simultaneously) the x component of the spin of one of the particles and the y components of the spins of the other two. What are the possible outcomes of the measurement? If instead all three x components are measured, what are the possible outcomes?

$$\left[\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}1 \right]$$

Quantum Mechanics *Problem 2*

A particle of mass m is incident with momentum $\hbar\mathbf{k}$ on a heavy scatterer. The scatterer is in its first excited state, with energy Δ above its ground state. The particle either scatters elastically, or the scatterer makes a transition to its ground state and the particle scatters 'inelastically', emerging with increased energy $\frac{\hbar^2 k^2}{2m} + \Delta$. All relativistic and recoil effects are negligible, and both states of the scatterer have zero angular momentum.

The process is described by a two-component wavefunction $\begin{bmatrix} \psi_1(x) \\ \psi_0(x) \end{bmatrix}$, which is a solution of the time-*independent* Schrödinger equation. $\psi_1(x)$ is the wavefunction of the particle with the scatterer in its excited state, and $\psi_0(x)$ with the scatterer in its ground state.

(a) Write down appropriate asymptotic ($|\mathbf{x}| \to \infty$) forms of ψ_1 and ψ_0 to describe the incident plane wave plus scattered waves. [This will require unknown functions of angles.] Specialize to the case when $|\mathbf{k}|$ is so small that only the angular-momentum-zero part of the incident plane wave ($\ell = 0$ partial wave) interacts with the scatterer. Express the total elastic and inelastic cross sections in terms of the parameters of the asymptotic wavefunctions.

(b) Separate the angular-momentum-zero parts of ψ_1 and ψ_0 into incoming and outgoing spherical waves. Find the condition imposed on your parameters by the requirement that the net flux of particles into a large sphere must be zero. [Do *not* try to do this for the original wavefunction – only the $\ell = 0$ parts.] With k fixed, what is the maximum possible value of the inelastic cross section? When that value is reached, what is the value of the elastic cross section?

[*Note*: A wavefunction $\psi(x)$ give a flux $\frac{\hbar}{m} \operatorname{Im} \psi^* \nabla \psi$. The average over angles of $e^{i\mathbf{k}\cdot\mathbf{x}}$ is $\frac{\sin kr}{kr}$; $k = |\mathbf{k}|$, $r = |\mathbf{x}|$. Note the distinction between the two separations, *incident* plus *scattered* or *incoming* plus *outgoing*.]