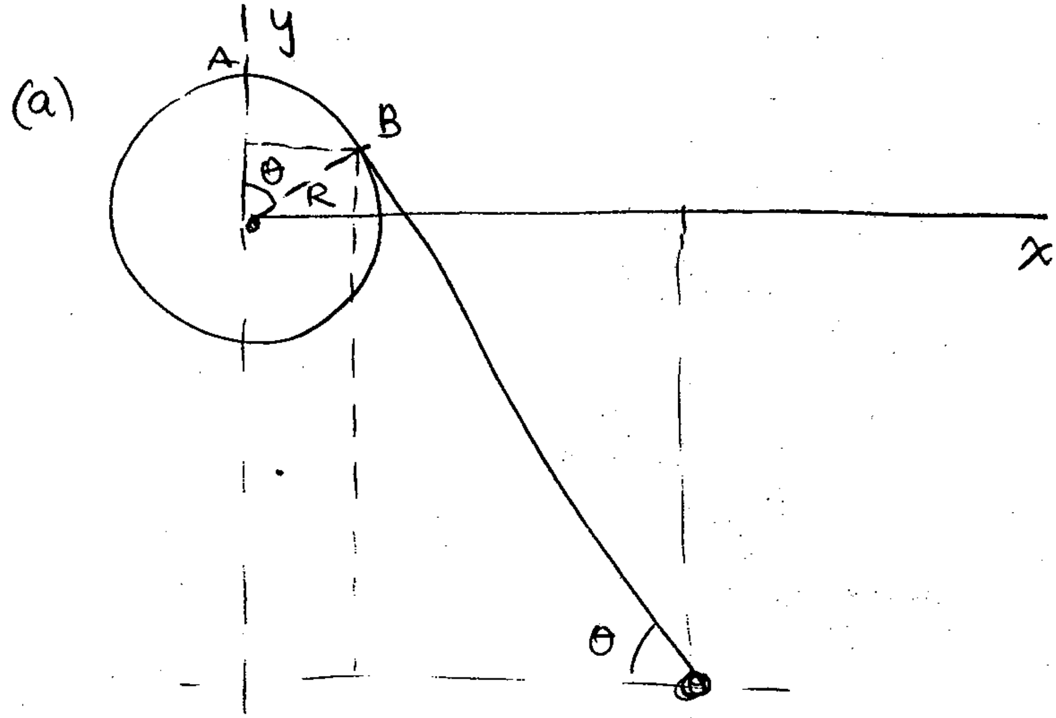
Part II – Spr '99

Problem 1 Mechanics

(a)



$$x = R\sin\theta + s\cos\theta$$
$$y = R\cos\theta - s\sin\theta$$

and $\quad s + R\theta = \ell \longrightarrow \dot{s} + R\dot{\theta} = 0$

$$\boxed{\dot{\theta} = -\frac{\dot{s}}{R}}$$

$$\dot{x} = (R\cancel{\cos}\theta - s\sin\theta)\dot{\theta} + \dot{s}\cancel{\cos}\theta$$
$$= \frac{s\dot{s}}{R}\sin\theta$$

$$\dot{y} = -(R\cancel{\sin}\theta + s\cos\theta)\dot{\theta} - \dot{s}\cancel{\sin}\theta$$
$$= \frac{s\dot{s}\cos\theta}{R}$$

$$\dot{x}^2 + \dot{y}^2 = \frac{s^2 \dot{s}^2}{R^2}$$

$$\boxed{L = \frac{1}{2} m \frac{s^2 \dot{s}^2}{R^2}}$$

(b) $$p = \frac{\partial L}{\partial \dot{s}} = \frac{m s^2 \dot{s}}{R^2} \rightarrow \dot{s} = \frac{p R^2}{m s^2}$$

$$H = p\dot{s} - L = \frac{1}{2} m \frac{s^2 \dot{s}^2}{R^2}$$

$$H = \frac{1}{2} m \frac{s^2}{R^2} \frac{p^2 R^4}{m^2 s^4}$$

$$\boxed{H = \frac{1}{2m} \frac{R^2}{s^2} p^2}$$

$$\frac{\partial H}{\partial p} = \boxed{\dot{s} = \frac{2H}{p}}$$

$$\frac{\partial H}{\partial s} = -\dot{p} = -\frac{2H}{s} \qquad \boxed{\dot{p} = \frac{2H}{s}}$$

Then

$$\frac{d}{dt}\left(\frac{p}{s}\right) = \frac{s\dot{p} - p\dot{s}}{s^2} = \frac{s\frac{2H}{s} - p\frac{2H}{p}}{s^2} = 0$$

$\left(\frac{p}{s}\right)$ is a constant of the motion

Indeed $H = E = \frac{R^2}{2m}\left(\frac{p}{s}\right)^2$

$$\boxed{\frac{p}{s} = \pm\frac{\sqrt{2mE}}{R}}$$

$$p\dot{s} = 2E$$

$$\pm\, s \cdot \frac{\cancel{\sqrt{2mE}}}{R}\,\dot{s} = \cancel{\sqrt{2E}}\,\sqrt{2E}$$

$$s\dot{s} = \pm\sqrt{\frac{2E}{m}}\,R$$

$$\frac{d}{dt}s^2 = \pm 2\sqrt{\frac{2E}{m}}\,R$$

$$\boxed{s^2(t) = \pm\sqrt{\frac{8E}{m}}\,R\,t + s_0^2}$$

$[Q_i, P_j] = \delta_{ij}$

(c) $Q = s^2$

$P = \dfrac{p}{\beta s}$

$$dQ \wedge dP = (2s\,ds) \wedge \frac{1}{\beta}\,\frac{s\,dp - p\cancel{ds}^{\,0}}{s^2}$$

$$= \frac{2s}{\beta} ds \wedge \frac{s\,dp}{s^2}$$

$$= \frac{2}{\beta}\, ds \wedge dp$$

need $\boxed{\beta = 2}$ $\boxed{\begin{array}{l} Q = s^2 \\ P = \dfrac{p}{2s} \end{array}}$

then $H = \dfrac{1}{2m} R^2 \left(\dfrac{p}{s}\right)^2 = \dfrac{1}{2m} R^2 (2P)^2$

$\boxed{H' = \dfrac{2}{m} R^2 P^2}$ New hamiltonian

With $H(p, q)$ HJ $\dfrac{\partial S}{\partial t} = -H\left(\dfrac{\partial S}{\partial q}, q\right)$

Thus we have

$$\boxed{\frac{\partial S}{\partial t} = -\frac{2}{m} R^2 \left(\frac{\partial S}{\partial Q}\right)^2}$$

$\boxed{S = S(Q, t)}$

$$S = W(Q) - \alpha t$$

$$-\alpha = -\frac{2}{m} R^2 \left(\frac{\partial W}{\partial Q}\right)^2$$

$$\left(\frac{dW}{\partial Q}\right)^2 = \frac{\alpha m}{2R^2} \rightarrow \boxed{\frac{dW}{dQ} = \pm\sqrt{\frac{\alpha m}{2}}\,\frac{1}{R}}$$

$$W(Q) = \pm\sqrt{\frac{\alpha m}{2}}\,\frac{Q}{R}$$

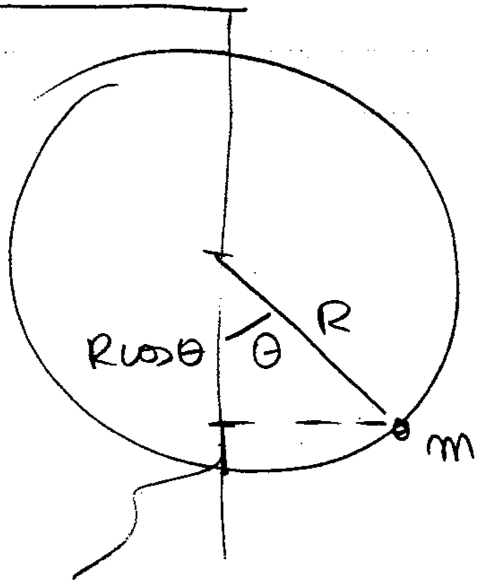
$$\boxed{S(Q,t,\alpha) = \pm\sqrt{\frac{\alpha m}{2}}\,\frac{Q}{R} - \alpha t}$$

$$\frac{\partial S}{\partial \alpha} = \beta = \text{constant}$$

$$\pm\frac{1}{2}\sqrt{\frac{m}{2\alpha}}\,\frac{Q}{R} - t = \beta$$

$$\boxed{Q = \pm\sqrt{\frac{8\alpha}{m}}\,R t \pm \sqrt{\frac{8\alpha}{m}}\,R\beta}$$

Mechanics 2



$R(1-\cos\theta)$

$$\vec{v} = R\dot{\theta}\,\vec{e}_\theta + \omega R\sin\theta\,\vec{e}_\phi$$

(a) $L = T - V$

$$= \frac{1}{2}m\left(R\dot{\theta}\,\vec{e}_\theta + \omega R\sin\theta\,\vec{e}_\phi\right)^2 - mgR(1-\cos\theta)$$

since constants are irrelevant (in the V term)

$$\boxed{L = \frac{1}{2}m\left(R^2\dot{\theta}^2 + \omega^2R^2\sin^2\theta\right) + mgR\cos\theta}$$ ✓

$$L = \frac{1}{2}mR^2\dot{\theta}^2 + \underbrace{\frac{1}{2}m\omega^2R^2\sin^2\theta + mgR\cos\theta}_{-V_{eff}}$$

$$\boxed{V_{eff} = -\frac{1}{2}mR^2\omega^2\sin^2\theta - mgR\cos\theta}$$ ✓

(b) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = \frac{\partial L}{\partial \theta}$$

$$\frac{1}{2} mR^2 \cdot 2\ddot\theta = m\omega^2R^2 \sin\theta\cos\theta - mgR\sin\theta$$

$$mR^2\ddot\theta = -mgR\sin\theta + m\omega^2R^2\sin\theta\cos\theta$$

$$\boxed{\ddot\theta = -\frac{g}{R}\sin\theta + \omega^2\sin\theta\cos\theta}$$ ✓

(c) $\theta = \text{constant} \rightarrow \dot\theta = 0, \rightarrow \ddot\theta = 0$

$$-\frac{g}{R}\sin\theta + \omega^2\sin\theta\cos\theta = 0$$

$$\sin\theta\left(\omega^2\cos\theta - \frac{g}{R}\right) = 0$$

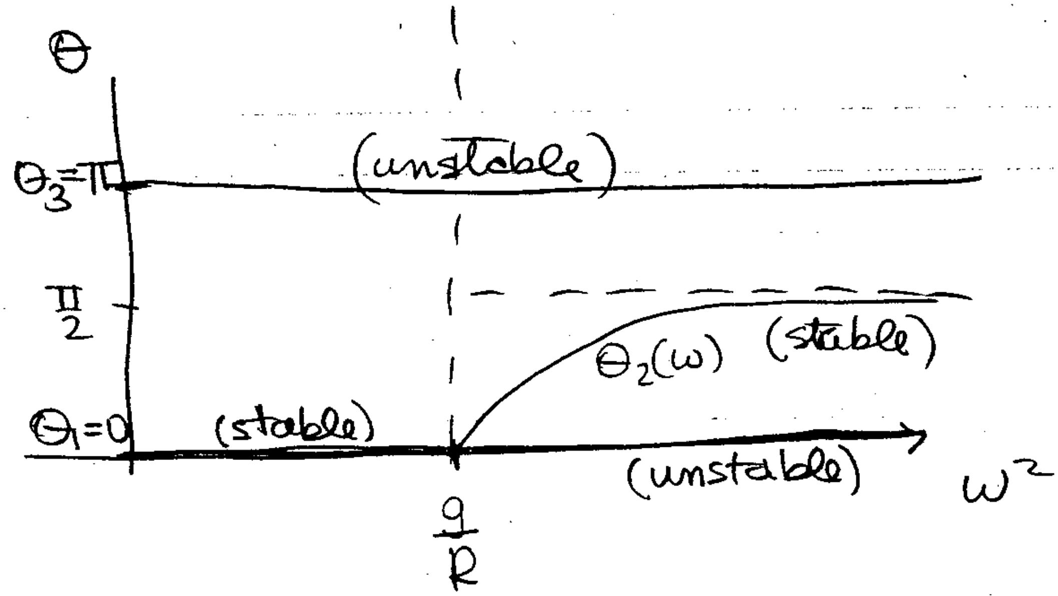
Three solutions

$$\boxed{\begin{aligned} \theta_1 &= 0 \\ \theta_2 &= \cos^{-1}\left(\frac{g}{R\omega^2}\right) \\ \theta_3 &= \pi \end{aligned}}$$

θ_2 requires $\omega^2 > \frac{g}{R}$

So 2 θ's for $\omega^2 < \frac{g}{R}$

3 θ's for $\omega^2 > \frac{g}{R}$



(d) $$\frac{1}{2} m R^2 \, 2\ddot{\Theta} = -\frac{\partial V_{eff}}{\partial \Theta}$$

$$\boxed{m R^2 \ddot{\Theta} = -\frac{\partial V_{eff}}{\partial \Theta}}$$

Equilibrium points Θ_i satisfy $\left.\frac{\partial V_{eff}}{\partial \Theta}\right|_{\Theta_i} = 0$

So for $\Theta = \Theta_i + \epsilon$ we get

$$m R^2 \ddot{\epsilon} = -\left\{ \left.\frac{\partial V}{\partial \Theta}\right|_{\Theta_i} + \epsilon \left.\frac{\partial^2 V}{\partial \Theta^2}\right|_{\Theta_i} \right\}$$

$$\boxed{m R^2 \ddot{\epsilon} = -\epsilon \left.\frac{\partial^2 V}{\partial \Theta^2}\right|_{\Theta_i}}$$

$$\Rightarrow \Omega_i^2 \equiv \frac{1}{mR^2} \left.\frac{\partial^2 V}{\partial \Theta^2}\right|_{\Theta_i}$$ for stability $\Omega_i^2 > 0$

oscillation frequency

$$\frac{\partial V}{\partial \theta} = -\frac{1}{2} m R^2 \omega^2 \, 2 \sin\theta \cos\theta + mgR\sin\theta$$

$$= -mR^2\omega^2 \sin\theta\cos\theta + mgR\sin\theta$$

$$\frac{\partial^2 V}{\partial \theta^2} = -mR^2\omega^2\left(\cos^2\theta - \sin^2\theta\right) + mgR\cos\theta$$

$$= -mR^2\omega^2\left(2\cos^2\theta - 1\right) + mgR\cos\theta$$

$$\boxed{\frac{1}{mR^2}\frac{\partial^2 V}{\partial \theta^2}\bigg|_{\theta_i} = -\omega^2\left(2\cos^2\theta_i - 1\right) + \frac{g}{R}\cos\theta_i}$$

1) $\theta_1 = 0$ $\qquad \Omega_1^2 = -\omega^2(2-1) + \frac{g}{R}$

$$\boxed{\Omega_1^2 = \frac{g}{R} - \omega^2}$$ ✓

stable for $\omega < \sqrt{\frac{g}{R}} = \omega_0$

unstable for $\omega > \sqrt{\frac{g}{R}} = \omega_0$

2) $\theta_2 = \cos^{-1}\frac{g}{R\omega^2}$ $\qquad \cos\theta_2 = \frac{g}{R\omega^2}$

$$\Omega_2^2 = -\omega^2\left(2\frac{g^2}{R^2\omega^4} - 1\right) + \frac{g^2}{R^2\omega^2}$$

$$= \omega^2 - \frac{g^2}{R^2\omega^2}$$

$$\Omega_2^2 = \omega^2 - \frac{g^2}{R^2\omega^2}$$

Stable when $\omega^2 > \frac{g^2}{R^2\omega^2}$ or $\omega^2 > \frac{g}{R}$

$\longrightarrow$ stable when it exists!

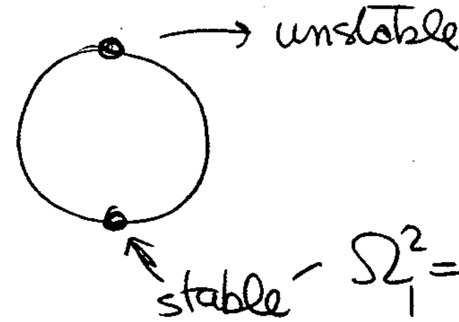
3) $\Theta_3 = \pi$ $\quad \Omega_3^2 = -\omega^2(2-1) - \frac{g}{R}$

$$\Omega_3^2 = -\omega^2 - \frac{g}{R}$$

Always unstable, no oscillations

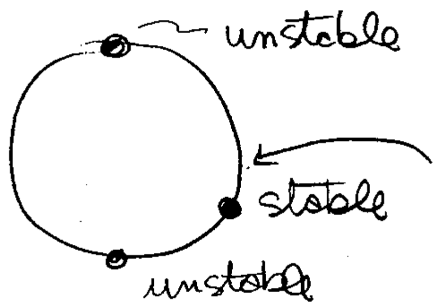
Regions of stability and instability indicated in the figure (p.3).

$\omega^2 < \frac{g}{R}$



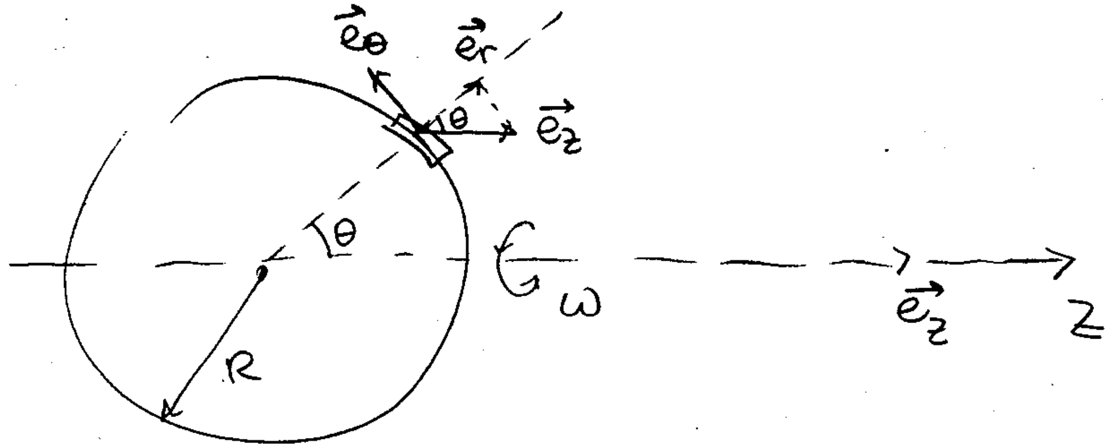
$\Omega_1^2 = \frac{g}{R} - \omega^2$

$\omega^2 > \frac{g}{R}$



$\Omega_2^2 = \omega^2 - \frac{g^2}{R^2\omega^2}$

Electromagnetism Problem 1



(a) $\vec{B}_{in} = B_0 \vec{e}_z = B_0(\vec{e}_r \cos\theta - \vec{e}_\theta \sin\theta)$

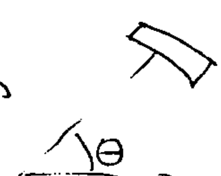
$$\vec{B}_{out} = \frac{2m\cos\theta}{r^3}\vec{e}_r + \frac{m\sin\theta}{r^3}\vec{e}_\theta$$

Continuity of the radial field

$$\frac{2m\cos\theta}{R^3} = B_0\cos\theta$$

$$\boxed{B_0 = \frac{2m}{R^3}}$$

Current K per unit length



$$\vec{K} = \sigma v(\theta)\,\vec{e}_\phi$$
$$= \frac{Q}{4\pi R^2}(R\sin\theta)\omega = \frac{Q\omega}{4\pi R}\sin\theta\,\vec{e}_\phi$$
$$= K_\phi\,\vec{e}_\phi$$

Discontinuity across the current sheet:

$$B_\theta(\text{out}) - B_\theta(\text{in}) = \frac{4\pi}{c} K_\phi$$

$$\rightarrow \quad \frac{m\sin\theta}{R^3} + B_0 \sin\theta = \frac{Q\omega}{cR}\sin\theta$$

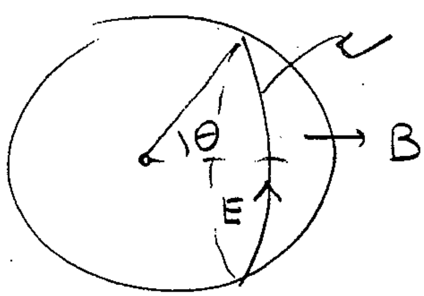
$$\frac{m}{R^3} + \frac{2m}{R^3} = \frac{Q\omega}{cR}$$

$$\frac{3m}{R^2} = Q\frac{\omega}{c} \rightarrow \boxed{m = \frac{1}{3}\left(\frac{\omega R}{c}\right) RQ} \checkmark$$

$$\boxed{m = \frac{1}{3}\left(\frac{\omega Q}{c}\right) R^2} \checkmark$$

$$\boxed{B_0 = \frac{2}{3}\frac{\omega}{c}\frac{Q}{R}} \checkmark$$

(b)



Apply Faraday's law to this loop

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$$

$$\oint E \cdot d\ell = -\frac{1}{c}\frac{\partial}{\partial t}(B \cdot \text{Area})$$

$$2\pi |E_\phi| R\sin\theta = \frac{1}{c}\left(\frac{dB_0}{dt}\right)\pi R^2 \sin^2\theta$$

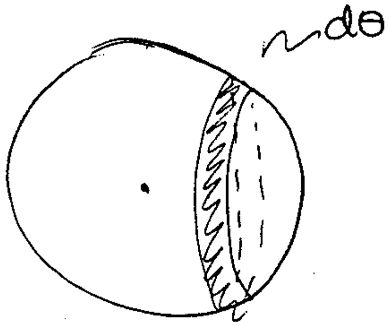
$$\therefore |E_\phi| = \frac{1}{2c}\frac{dB_0}{dt} R\sin\theta$$

$$= \frac{1}{2c}\frac{2}{3}\frac{\dot{\omega}}{c}\frac{Q}{\cancel{R}}\cancel{R}\sin\theta$$

$$\boxed{|E_\phi| = \frac{1}{3}\frac{Q\sin\theta}{c^2}\dot{\omega}} \checkmark$$

— sign

(c) The torque tends to slow down the ball



$$dq = \frac{Q}{4\pi R^2}\, da = \frac{Q}{4\pi R^2}\,(R\,d\theta)\,2\pi R\,\sin\theta$$

$$= \frac{Q}{4\pi}\,(2\pi)\sin\theta\, d\theta$$

$$dq = \frac{Q}{2}\sin\theta\, d\theta$$

$$dF = E_\phi(\theta)\, dq$$

$$= \frac{1}{3}\,\frac{Q\sin\theta}{c^2}\,\dot{\omega}\;\; \frac{Q}{2}\sin\theta\, d\theta$$

$$dF = \frac{1}{6}\,\frac{Q^2\dot{\omega}}{c^2}\sin^2\theta\, d\theta$$

$$dG = dF \cdot R\sin\theta$$

$$\boxed{dG = \frac{1}{6}\,\frac{Q^2\dot{\omega}}{c^2}\,R\sin^3\theta\, d\theta}$$

$$\int d\theta\, \sin^3\theta = \frac{4}{3}$$

$$\boxed{\vec{G} = -\frac{2}{9}\,\frac{Q^2 R}{c^2}\,\dot{\omega}\,\vec{e}_z}$$

I ✓

(d) $$\frac{d\vec{L}}{dt} = \vec{G}$$

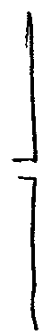
$$\frac{d(I\omega)}{dt} = G_{ext} - \frac{2}{9}\frac{Q^2R}{c^2}\dot{\omega}$$

$$I_{mech}\,\dot{\omega} + \frac{2}{9}\frac{Q^2R}{c^2}\dot{\omega} = G_{ext}$$

so we read:

$$\boxed{I_{mag} = \frac{2}{9}\frac{Q^2R}{c^2}}$$ ✓

Problem 2 E & M



a) $$I(z,t) = I_0\left(1 - \frac{\alpha_0 |z|}{d}\right)\cos\omega t$$

For $|z| = \frac{d}{2}$ need $I(z,t) = 0$

$$\rightarrow \boxed{\alpha_0 = 2}$$

$$I(z,t) = \begin{cases} I_0\left(1 - \frac{2z}{d}\right)\cos\omega t & z>0 \\ I_0\left(1 + \frac{\alpha_0 z}{d}\right)\cos\omega t & z<0 \end{cases}$$

Charge conservation $\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0$

Here
$$\frac{dI}{dz} + \frac{d\lambda(z,t)}{dt} = 0$$

$$\frac{d\lambda}{dt}(z,t) = -\frac{dI}{dz} = \begin{cases} \frac{2I_0}{d}\, e^{-i\omega t} \\ -\frac{2I_0}{d}\, e^{-i\omega t} \end{cases}$$

$$\lambda(z) = \frac{2I_0}{-i\omega d} \qquad z>0$$

$$- \frac{2I_0}{-i\omega d} \qquad z<0$$

$$\boxed{\lambda(z) = \pm \frac{2iI_0}{\omega d}} \quad \begin{matrix} z>0 \\ z<0 \end{matrix}$$

(b) $$\vec{p} = p_z e_z$$

$$p_z = \int_{-\frac{d}{2}}^{\frac{d}{2}} z\,\lambda(z)\,dz$$

$$= 2\int_0^{d/2} z\left(\frac{2iI_0}{\omega d}\right)dz$$

$$\vec{p} = \frac{4iI_0}{\omega d}\cdot\frac{z^2}{2}\Big|^{d/2} = \frac{iI_0 d}{2\omega}\vec{e}_z$$

$$\boxed{\vec{p} = \frac{iI_0 d}{2\omega}\vec{e}_z} \qquad \frac{\omega}{c} = k$$

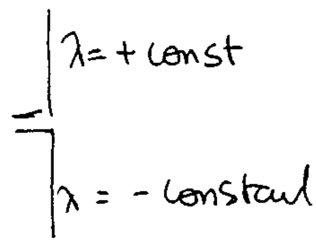
$$P_{\text{center fed}} = \frac{ck^4}{3}|\vec{p}|^2 = \frac{ck^4}{3}\frac{I_0^2 d^2}{4c^2k^2}$$

$$\boxed{P_{\text{center-fed}} = \frac{I_0^2 (kd)^2}{12c}}$$

(c) No magnetic dipole term since ✓

$\vec{x} \times \vec{J} = 0$ on the antenna (origin at ⊥)

No electric quadrupole since the charge distribution



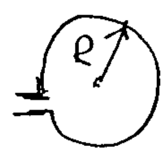
has no quadrupole term

$$Q_{\alpha\beta} = \int (3x_\alpha x_\beta - r^2 \delta_{\alpha\beta})\, \rho(x)\, d^3x$$

only $z \neq 0$ $\int r^2 \rho(x) = 0$ by symmetry

and $\int z^2 \rho(x) = 0$ ✓

(d) Now wrap the wire



$2\pi R = d$ $\quad R = \dfrac{d}{2\pi}$

Area $\pi R^2 = \pi \dfrac{d^2}{4\pi^2} = \dfrac{d^2}{4\pi}$

$$\boxed{m = \frac{I_0 d^2}{4\pi c}}$$

Power, circle $= \dfrac{c k^4}{3} \dfrac{I_0^2 d^4}{16\pi^2 c^2}$

$$\boxed{P_{circle} = \frac{I_0\,(kd)^4}{(4\pi^2)\,12\,c}}$$

$$\frac{P_{circle}}{P_{center\text{-}fed}} = \frac{(kd)^4}{(kd)^2}\,\frac{1}{(2\pi)^2}$$

$$\frac{P_{circle}}{P_{center\text{-}fed}} = \left(\frac{1}{2\pi}\,kd\right)^2 = \left(\frac{d}{\lambda}\right)^2$$

$$\boxed{\frac{P_{circ}}{P_{center\text{-}fed}} = \left(\frac{d}{\lambda}\right)^2}$$

It is indeed very small. Electric dipole is leading term in radiation.

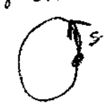
(e) In this case $\rho(\theta) = 0$ NO! since, $I(\theta)$ is constant. So there is no dipole nor quadrupole term.

$$\rho \propto \frac{1}{r^2}\,\delta\!\left(r - \frac{d}{2\pi}\right)\delta\!\left(\theta - \frac{\pi}{2}\right)$$

$\Rightarrow$ Dipole

← this is wrong,

$\frac{dI}{ds} = 0$ along circle, no charge, no moments



once you give a finite thickness to the wire there is no δ. This is mathematical idealization.

Solution to SM1.

i) $$e^{-\frac{F}{kT}} = \sum e^{-\frac{E}{kT}} = \prod_{r=1}^{\infty} \frac{1}{1-e^{-\frac{\hbar\omega_0}{kT}r}}$$

$$F = kT\sum_{r=1}^{\infty} \ln\left(1-e^{-\frac{\hbar\omega_0}{kT}r}\right)$$

$$\approx kT\int_0^{\infty} \ln\left(1-e^{-x}\right)\frac{kT}{\hbar\omega_0}dx$$

$$\text{Integral} = -\int_0^{\infty}\left(e^{-x}+\frac{e^{-2x}}{2}+\dots\right)dx = -\frac{\pi^2}{6}$$

$$\text{so } F = -\frac{\pi^2}{6}\frac{(kT)^2}{\hbar\omega_0}$$

$$S = -\frac{dF}{dT} = \frac{\pi^2}{3}k\frac{kT}{\hbar\omega_0}$$

$$E = F+TS = +\frac{\pi^2}{6}\frac{(kT)^2}{\hbar\omega_0}$$

$$\sqrt{\frac{E}{\hbar\omega_0}} = \frac{\pi}{\sqrt{6}}\frac{kT}{\hbar\omega_0}, \quad S = \pi\sqrt{\frac{2}{3}}\,k\sqrt{\frac{E}{\hbar\omega_0}}$$

Now identify S_N with $k\ln p(N)$ and E_N with $\hbar\omega_0 N$

which gives $p(N) \sim e^{\pi\sqrt{\frac{2N}{3}}}$. We would not expect to get anything beyond the exponent of the H-R formula this way, without ~~the~~ calculating fluctuations in the canonical ensemble (at least)

b) This time $\frac{E}{\hbar\omega_0} = \sqrt{N}$, $S = k \ln p(N)$
$= k\pi\sqrt{\frac{2N}{3}}$

so $S = \pi\sqrt{\frac{2}{3}}\, k \frac{E}{\hbar\omega_0}$

$\frac{dS}{dE} = \frac{1}{T}$ so $T = T_0 = \frac{1}{\pi}\sqrt{\frac{3}{2}}\frac{\hbar\omega_0}{k}$, independent of E.

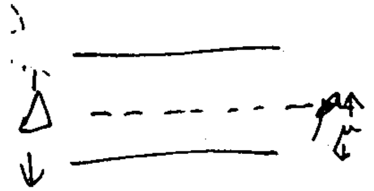
This is a system with a density of states $\sim e^{E/kT_0}$.

~~Clearly the canonical ensemble is only defined for~~ ~~$\frac{1}{kT}$~~ ~~[illegible]~~

~~ie $T < T_0$.~~

If we used it as a heat bath in contact with a smaller system, the small system will always go into a canonical ensemble at temperature T_0.

Solution to S.M.2



Assume the chemical potential is in the gap

Number in valence band is $\dfrac{N}{e^{-\frac{\mu}{T}}+1} = N - \dfrac{N}{e^{\frac{\mu}{T}}+1}$

Number in conduction band is $\dfrac{N}{e^{\frac{\Delta-\mu}{T}}+1}$

so μ must be halfway between bands (# of holes = # of excited electrons)

and $N_{cond} = \dfrac{N}{e^{\frac{\Delta}{2T}}+1}$, $N_{val} = \dfrac{N}{e^{-\frac{\Delta}{2T}}+1}$

) Rate of upward transitions is proportional to $N_{valence}$, and to $N - N_{conduction}$ i.e. to $N^2_{valence}$, and to the Bose-Einstein ~~occupation~~ factor $\dfrac{1}{e^{\frac{\Delta}{T}}-1}$ for the number of photons, energy Δ available to be absorbed.

Rate of downward transitions is proportional to N_{cond} and $N - N_{val}$ i.e. N^2_{cond} and to the factor $1 + \dfrac{1}{e^{\frac{\Delta}{T}}-1} = \dfrac{e^{\frac{\Delta}{T}}}{e^{\frac{\Delta}{T}}-1}$ for spontaneous plus stimulated emission.

In Equilibrium $N^2_{cond}\, e^{\frac{\Delta}{T}} = N^2_{val}$, $\dfrac{N_{cond}}{N_{val}} = e^{-\frac{\Delta}{2T}}$ ✓

[Constants in rates ↑ and ↓ other than those described above are same in each direction]

c) Now get $N_{cond} = \frac{N}{\Delta}\int_{\Delta}^{2\Delta} \frac{dE}{e^{\frac{E-\mu}{T}}+1} = N - N_{val}$

$$= \frac{N}{e^{\frac{\mu}{T}}+1}$$

Assume $\frac{\Delta}{T} \gg 1$ and μ in gap. Then approximate by

$$\frac{N}{\Delta}\int_{\Delta}^{2\Delta} dE\, e^{\frac{\mu-E}{T}} = Ne^{-\frac{\mu}{T}}$$

$$e^{-\frac{2\mu}{T}} = \frac{T}{\Delta}\left(e^{-\frac{\Delta}{T}} - \cancel{e^{-\frac{2\Delta}{T}}}\right) \quad \left[e^{\frac{2\mu-\Delta}{T}} = \frac{\Delta}{T}\right.$$

$$\left. 2\mu = \Delta\left(1+\frac{T}{\Delta}\ln\frac{\Delta}{T}\right) \quad \text{so as } \frac{T}{\Delta}\to 0, \quad \mu \to \frac{\Delta}{2} \right]$$

$$N_{cond} = Ne^{-\frac{\mu}{T}} = Ne^{-\frac{\Delta}{2T}}\sqrt{\frac{T}{\Delta}}$$

[Calculation in brackets only needed to check that μ is in middle of gap]

QM1. Solution

a) $|\psi\rangle$ is sum of states with $S_z^{tot} = \pm\frac{3}{2}\hbar$. Allowed values of S $(S^2 = s(s+1)\hbar^2)$ are $\frac{1}{2}, \frac{3}{2}$.

$S_z = \pm\frac{3}{2}$ can only belong to $S = \frac{3}{2}$ to $(S^{tot})^2|\psi\rangle = \frac{15}{4}\hbar^2|\psi\rangle$

b) $S_x + iS_y$ increases S_z^{tot} by 1

so R increases S_z^{tot} by 3. Only non-zero matrix element is $\langle S_z^{tot} = \frac{3}{2}|R|S_z^{tot} = -\frac{3}{2}\rangle$

i.e $\langle\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}|R|-\frac{1}{2}\,-\frac{1}{2}\,-\frac{1}{2}\rangle$

Thus $R|-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}\rangle = a|\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\rangle$

Since $(S_x^{(1)} + iS_y^{(1)})|-\frac{1}{2}\rangle = |+\frac{1}{2}\rangle$ and there are 6 terms in R that send $|---\rangle$ to $|+++\rangle$, $a = 6$

In the same way $R^\dagger|\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\rangle = 6|-\frac{1}{2}\,-\frac{1}{2}\,-\frac{1}{2}\rangle$

and so $R|\psi\rangle = -6|\psi\rangle$

Clearly $R(|+++\rangle + |---\rangle) = +6|\psi\rangle$

and R on all other states (6) is zero.

:) $AB = (\sigma_x\sigma_y)^{(1)}(\sigma_y\sigma_x)^{(2)}(\sigma_y\sigma_y)^{(3)}$

$BA = (\sigma_y\sigma_x)^{(1)}(\sigma_x\sigma_y)^{(2)}(\sigma_y\sigma_y)^{(3)}$

2 pairs anticommute so $AB = BA$

and in same way $AC = CA$ ~~$AD = DA$~~ $BC = CB$

$AD = (\sigma_x\sigma_x)^{(1)}(\sigma_y\sigma_x)^{(2)}(\sigma_y\sigma_x)^{(3)}$

$DA = (\sigma_x\sigma_x)^{(1)}(\sigma_x\sigma_y)^{(2)}(\sigma_x\sigma_y)^{(3)}$

Again 2 pairs anticommute so $AD = DA$

$ABCD = (\sigma_x\sigma_y\sigma_y\sigma_x)^{(1)}(\sigma_y\sigma_x\sigma_y\sigma_x)^{(2)}(\sigma_y\sigma_y\sigma_x\sigma_x)^{(3)}$

$= 1^{(1)}(-1)^{(2)}(1)^{(3)} = -1$

d) $\sigma_x|\pm\tfrac{1}{2}\rangle = |\mp\tfrac{1}{2}\rangle$ $\quad$ $\sigma_y|\pm\tfrac{1}{2}\rangle = \pm i\,|\mp\tfrac{1}{2}\rangle$

so $A|+++\rangle = -|---\rangle$ $\quad$ $A|\psi\rangle = |\psi\rangle$

$A|---\rangle = -|+++\rangle$

and $B|\psi\rangle = |\psi\rangle$ $\quad$ $C|\psi\rangle = |\psi\rangle$

$D|+++\rangle = |---\rangle$ $\quad$ so $D|\psi\rangle = -|\psi\rangle$

$D|---\rangle = |+++\rangle$

(agrees with $\lambda = -1$ above)

e) Measure $S_x^{(1)} S_y^{(2)} S_y^{(3)}$. Because $A = +1$,

results must be $\pm\frac{1}{2}\hbar, \pm\frac{1}{2}\hbar, \pm\frac{1}{2}\hbar$ with signs

$$\begin{cases} + + + \\ + - - \\ - + - \\ - - + \end{cases}$$

Measure $S_x^{(1)} S_x^{(2)} S_x^{(3)}$ must get signs

$$\begin{cases} - + + \\ + - + \\ + + - \\ - - - \end{cases}$$

Solution to QM2

a) $$\psi_1(\vec{x}) \sim e^{i\vec{k}\cdot\vec{x}} + f_1(\hat{x})\frac{e^{ikr}}{r}$$

$$\psi_0(\vec{x}) \sim f_0(\hat{x})\frac{e^{ik'r}}{r} \qquad \frac{\hbar^2 k'^2}{2m} = \frac{\hbar^2 k^2}{2m} + \Delta$$

$l=0$ scattering only, since scatter has no angular momentum, f_1 and f_0 are both independent of $\hat{x}$.

Plane wave has flux $\frac{\hbar k}{m}$ per unit area,

f_1 corresponds to " $|f_1|^2 \frac{\hbar k}{m r^2}$ so total through sphere $\frac{\hbar k}{m} 4\pi |f_1|^2$

f_0 " " $|f_0|^2 \frac{\hbar k'}{m r^2}$, total $\frac{\hbar k'}{m} 4\pi |f_0|^2$

so $\sigma_{el} = 4\pi |f_1|^2 \qquad \sigma_{inel} = 4\pi |f_0|^2 \frac{k'}{k}$

b) $$\left(e^{i\vec{k}\cdot\vec{x}}\right)_{l=0} = \frac{\sin kr}{kr} = \frac{e^{ikr} - e^{-ikr}}{2ikr}$$

so $l=0$ parts are $$\psi_1 \sim -\frac{e^{-ikr}}{2ikr} + \frac{e^{ikr}}{r}\left(\frac{1}{2ik} + f_1\right)$$

$$\psi_0 \sim \frac{e^{ik'r}}{r} f_0$$

Clearly flux due to ψ_0 is $\frac{\hbar k'}{m}|f_0|^2 4\pi$ outwards.

For $\psi = A\frac{e^{ikr}}{r} + B\frac{e^{-ikr}}{r}$, $\psi^*\vec{\nabla}\psi = \left(A^* e^{-ikr} + B^* e^{ikr}\right) ik\left(Ae^{ikr} - Be^{-ikr}\right)\frac{\hat{r}}{r^2} + O\left(\frac{1}{r^3}\right)$

so $\operatorname{Im}\psi^* \vec{\nabla}\psi = \frac{\hat{x}}{r^2} k\left(|A|^2 - |B|^2\right)$ and flux $= \frac{\hbar k}{m} 4\pi\left(|A|^2 - |B|^2\right)$

So constraint is $k\left|\frac{1}{2ik} + f_1\right|^2 + k'|f_0|^2 = \frac{k}{4k^2}$

Clearly maximum σ_{inel} is when $f_1 = -\frac{1}{2ik}$

when $k'|f_0|^2 = \frac{1}{4k}$ and $\sigma_{inel} = \frac{\pi}{k^2}$.

In that case $\sigma_{el} = \frac{\pi}{k^2}$ also.