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Zhaoyi Li

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QM1

a)[1]

$$\langle\overline{\alpha}|\overline{\beta}\rangle = e^{-\frac{|\alpha|^2}{2}-\frac{|\beta|^2}{2}+\alpha^*\beta}$$

So the pure coherent states are never orthogonal for any choices of α and β.

b)

1

$$\begin{aligned}\langle\psi_\pm|\psi_\pm\rangle =&\frac{\mathcal{N}_\pm{}^2(\beta)}{2}(1+1\pm\langle\overline{\beta}|\overline{-\beta}\rangle\pm\langle\overline{-\beta}|\overline{\beta}\rangle)\\ =&\mathcal{N}_\pm{}^2(\beta)(1\pm e^{-2|\beta|^2})\end{aligned}$$

Note that $\langle\overline{\beta}|\overline{-\beta}\rangle = \langle\overline{-\beta}|\overline{\beta}\rangle$. Therefore $\mathcal{N}_\pm = \frac{1}{\sqrt{1\pm e^{-2|\beta|^2}}}$, clearly $\mathcal{N}_\pm(\infty) = 1$.

2

We verify that $\langle\psi_\pm|\psi_\mp\rangle = 1 - 1 + \langle\overline{\beta}|\overline{-\beta}\rangle - \langle\overline{\beta}|\overline{-\beta}\rangle = 0$.

3

$\langle n|\psi_\pm\rangle = \frac{1}{\sqrt{2n!(e^{|\beta|^2}-e^{-|\beta|^2})}}(\beta^n \pm (-\beta)^n)$, and

$$P_\pm(n) = \frac{e^{-|\beta|^2}}{2n!(1\pm e^{-2|\beta|^2})}\left(\beta^n \pm (-\beta)^n\right)\left((\beta^*)^n \pm (-\beta^*)^n\right) = \frac{e^{-|\beta|^2}}{n!(1\pm e^{-2|\beta|^2})}|\beta|^{2n}(1\pm(-1)^n)$$

[1] Note that the summation given in the hint should start from $n = 0$, i.e. $\Sigma_{n=0}^{\infty}\frac{x^n}{n!} = e^x$.

(c)

1

$$\begin{aligned}[K_1, K_2] =&\frac{1}{4i}([a^2, a^{\dagger 2}] - [a^{\dagger 2}, a^2]) \\ =&\frac{1}{2i}[a^2, a^{\dagger 2}] \\ =&-4iK_3\end{aligned}$$

2

By uncertainty principle we know $\Delta K_1 \Delta K_2 \geq |\frac{1}{2i}[K_1, K_2]| = |2\langle K_3 \rangle|$.

3

We calculate

$$\Delta^2 K_1 = \langle K^2 \rangle - \langle K \rangle^2 \tag{1}$$

$$= \frac{1}{2}(2|\gamma^2| + 1) \tag{2}$$

Following the same calculation we obtain the identical result for $\Delta^2 K_2$. Therefore $\Delta K_1 \Delta K_2 = |2\langle K_3 \rangle| = |\gamma|^2 + \frac{1}{2}$

QM2

a)

Note that the correct form of the wave equation should actually be $\psi(\vec{x}) = e^{\frac{i}{\hbar}(p_x x + p_z z)}\phi(y)\chi_{\pm}$. Plugging in the time-independent Schrödinger equation, we get $\frac{1}{2m}(\hat{p}_y^2 + p_z^2 + (p_x + \frac{q}{c}By)^2)\phi \mp \frac{\hbar q B}{2mc}\phi = E\phi$.

b)

Rearranging the Schrödinger equation, we immediately recognize the form of the equation as that of a 1D SHO with angular frequency $\omega = \frac{qB}{mc}$ and a horizontal shift of $-\frac{p_x c}{qB} = y_0$ to the right:

$$\left(\frac{\hat{p}_y^2}{2m} + \frac{1}{2}m\omega^2(y - y_0)^2\right)\phi = \left(E - \frac{p_z^2}{2m} \pm \frac{\hbar\omega}{2}\right)\phi$$

We can thus read off the spectra $E = \hbar\omega(n + \frac{1}{2}) + \frac{p_z^2}{2m} \mp \frac{\hbar\omega}{2} = \hbar\omega(n + \frac{1}{2} \mp \frac{1}{2}) + \frac{p_z^2}{2m}$.

c)

In the new gauge, the momentum terms are adjusted to $p_x + \frac{qBy}{c} - \frac{q}{c}\partial_x\Lambda$, $\hat{p}_y - \frac{q}{c}\partial_y\Lambda$, and $p_z - \frac{q}{c}\partial_z\Lambda$. However, upon evaluating these terms on the transformed wave function, ψ', the additional components involving Λ all cancel out, resulting in the original Schrödinger equation.

d)

Based on the given requirement, it is evident that a suitable choice for the vector potential is $Bx\hat{j}$, which can be obtained through the gauge transformation $(By, Bx, 0)$ from the original vector potential, i.e. $\Lambda(\vec{x}) = Bxy$. In this scenario, the momentum term in the Schrödinger equation simplifies to $\frac{1}{2m}(\hat{p}_x^2 + p_z^2 + \frac{q^2B^2}{c^2}(x - x_0)^2)$, where $x_0 = \frac{p_y c}{qB}$. And the corresponding wavefunctions are given by

$$\psi'(\vec{x}) = e^{\frac{i}{\hbar}(p_y y + p_z z)}\phi'(x)\chi_{\pm}$$

Consequently, we observe an oscillator-like behavior that is confined to the x direction but may exhibit shifts in the y direction.