Jan 27, 2021 QM

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Monday 8th January, 2024

QM1

a)

First, we work out the force for both cases $r < R$ and $r \geq R$:

$$F = \begin{cases} -\dfrac{rZe^2}{R^3} & r < R \\ -\dfrac{Ze^2}{r^2} & r \geq R \end{cases}$$

To get the potential energy for $r \geq R$, we simply compute $V_{\text{out}} = -\int_\infty^r F(\mathbf{r}) \cdot d\mathbf{r} = -\frac{Ze^2}{r}$, where we took the reference point $V(\infty) = 0$. For $r < R$, $V_{\text{in}} = -\int_\infty^r F(\mathbf{r}) \cdot d\mathbf{r} = -\int_\infty^R F_{\text{out}}(\mathbf{r}) \cdot d\mathbf{r} - \int_R^r F(\mathbf{r}) \cdot d\mathbf{r} = -\frac{Ze^2}{R} + \frac{Ze^2}{2R^3}(r^2 - R^2)$. In summary, we have:

$$V = \begin{cases} \dfrac{Ze^2}{2R}\left(\dfrac{r^2}{R^2} - 3\right) & r < R \\ -\dfrac{Ze^2}{r} & r \geq R \end{cases}$$

And we add the kinetic energy term, $\mathbf{p}^2/2m_\tau$, to V to get the Hamiltonian for the τ.

b)

When τ is confined inside the nucleus, the problem reduces to that of a 3D simple harmonic oscillator, where we view the r^2 term in V_{in} as $\frac{1}{2}m_\tau\omega^2 r^2$. Therefore, the frequency is given by $\omega = \sqrt{\frac{Ze^2}{m_\tau R^3}}$. The energy is given by $\omega\hbar(n+\frac{3}{2}) - \frac{3}{2}\frac{Ze^2}{R}$. The first and second energy levels are $|0\rangle$ (spin singlet) and $a_+^\dagger|0\rangle, a_3^\dagger|0\rangle, a_-^\dagger|0\rangle$, where $a_\pm = \frac{1}{\sqrt{2}}(ia_y \mp a_x)$, and it is possible to verify these operators forms a commuting set. In other words, the first excited state has triple degeneracy, and therefore it has to be a spin triplet: we can relate the z angular momentum to the ladder operators via the relation $J_z = \hbar(N_R - N_L)$. These three states are all diagonalized and non-degenerate under J_z. Thus the total angular momentum is $\sqrt{\hbar^2 j(j+1)} = 0$ for the ground state and $\sqrt{2}\hbar$ for the excited state.

c)

After perusing the question it seems to me that we are asked to evaluate the perturbation treating V as an SHO potential. To do so, we evaluate directly

$$\begin{aligned}\langle \mathbf{p}^4 \rangle =& \frac{\hbar^4}{4d^4}\langle(((a_x^\dagger - a_x)^2 + (x \leftrightarrow y \leftrightarrow z))^2\rangle \\ =& \frac{\hbar^4}{4d^4}((1+1+1)^3 + 2+2+2) \\ =& \frac{15\hbar^4}{4d^4}\end{aligned}$$

Note that from the first line to the second line we observe the fact that the $aa^\dagger$ terms in the expansion contribute 1 to the final result, and the $aaa^\dagger a^\dagger$ contribute 2, therefore $\Delta E = -\frac{15}{32}\frac{(\hbar\omega)^2}{mc^2}$

d)

$$\Delta V = V - V' = Ze^2((\frac{1}{r} - \frac{1}{R}) + \frac{1}{2R^2}(r^2 - R^2)).$$

When this energy difference is treated as a perturbation, there is $E^{(1)} = \langle \Delta V \rangle = \frac{Ze^2}{2R}\langle 000 | (\frac{r^2}{R^2} + \frac{2R}{r} - 3) | 000 \rangle$.

QM2

a)

First, we note that a generic wavefunction of this space could be represented as $\sum_\sigma \psi_\sigma |\sigma\rangle$, where σ is the spin index. We therefore represent $\psi_\sigma(x)$ as a two component spinor $\psi(x) = \begin{pmatrix} \psi_0(x) \\ \psi_1(x) \end{pmatrix}$. Since the incoming particle is prepared in the lower-energy state, we know that only the second component, which corresponds to $|1\rangle$, exists for the incoming wave. When $x < 0$, ψ attains a plane wave solution, i.e. $\psi_{x<0} = \begin{pmatrix} B_0 e^{\frac{-ixp_0}{\hbar}} \\ B_1 e^{\frac{-ixp_1}{\hbar}} + Ae^{\frac{ixp_1}{\hbar}} \end{pmatrix}$, where A is the amplitude of the incoming wave. For $x > 0$, we can set our wavefunction to be the similar form $\psi_{x>0} = \begin{pmatrix} C_0 e^{\frac{ixp_0}{\hbar}} \\ C_1 e^{\frac{ixp_1}{\hbar}} \end{pmatrix}$. Note that here WLOG we can take the left-propagating wave to be 0.

b)

In the usual 1-D case, we know that for δ-function potentials $V = \alpha\delta(x)$, where $\alpha = u\sigma_x$, there is the relation $(\Delta\psi')_{x=0} = \frac{2m\alpha}{\hbar^2}\psi(0)$ between ψ and its derivative (with respect to x) at $x = 0$. For the current case, there is the similar relation

$$\frac{2mu}{\hbar^2}\psi_{1t}(0) = \psi'_{0t}(0) - \psi'_{0r}(0) - \psi'_{0i}(0)$$
$$\frac{2mu}{\hbar^2}\psi_{0t}(0) = \psi'_{1t}(0) - \psi'_{1i}(0)$$

Plugging in our ansatz, we obtain the following equations

$$\frac{2mu}{\hbar^2}C_0 = \frac{ip_1}{\hbar}(C_1 + B_1 - A)$$
$$\frac{2mu}{\hbar^2}C_1 = \frac{ip_0}{\hbar}(C_0 + B_0)$$

where $p_1 = \sqrt{2mE_v}$. Here note that the momentum satisfies the relation $\frac{p_0^2}{2m} + \frac{c}{2} = \frac{p_1^2}{2m} - \frac{c}{2}$. Therefore, $p_0 = \sqrt{2m(E_v - c)}$ is well defined for $E_v > c$. Besides these two, there are two more continuity equations at $x = 0$:

$$B_0 = C_0$$
$$B_1 + A = C_1$$

c)

Solving these equations (we keep the parameter C_0 unsolved), we obtain $B_0 = C_0$, $C_1 = \frac{ip_0\hbar}{mu}C_0$, and $C_0 = \frac{ip_1\hbar}{mu}B_1$. From this, we have the total transmission coefficient

$$T = \frac{|C_0|^2 + |C_1|^2}{|A|^2} = \frac{p_1^2\hbar^2(p_0^2\hbar^2 + m^2u^2)}{(p_0p_1\hbar^2 + m^2u^2)^2} = \frac{2E_v m^2\hbar^2\left(2\hbar^2(E_v - c) + mu^2\right)}{\left(2m\hbar^2\sqrt{E_v(E_v - c)} + m^2u^2\right)^2}$$

d)

When $c > E_v$, we instead define $p_0 = \sqrt{2m(c - E_v)}$ (but p_1 stays the same) and the ansatz becomes $\psi_{x<0} = \begin{pmatrix} B_0 e^{\frac{xp_0}{\hbar}} \\ B_1 e^{\frac{-ixp_1}{\hbar}} + Ae^{\frac{ixp_1}{\hbar}} \end{pmatrix}$ and $\psi_{x>0} = \begin{pmatrix} C_0 e^{\frac{-xp_0}{\hbar}} \\ C_1 e^{\frac{ixp_1}{\hbar}} \end{pmatrix}$. Note that the higher energy state, corresponding to $|0\rangle$, is now real. Again, solving for the boundary conditions, we obtain $C_1 = \frac{-p_0\hbar}{mu} C_0$ and $A = \left(\frac{mui}{p_1\hbar} + \frac{p_0\hbar}{mu}\right) C_0$. The transmission coefficient is then

$$T = \frac{|C_1|^2}{|A|^2} = \frac{p_1^2 p_0^2 \hbar^4}{p_1^2 p_0^2 \hbar^4 + m^4 u^4} = \frac{4E_v \hbar^4 (c - E_v)}{4E_v \hbar^4 (c - E_v) + m^2 u^4}$$

Note that T only contains C_1 in the denominator because the wavefunction corresponding to $|0\rangle$ is static and has zero probability current.