Part 2 (1998): Problem 2 (Mechanics)

a)

$$\vec{\omega} = \omega(\sin\alpha\ \hat{n}_1 + \cos\alpha\ \hat{n}_3) \tag{1}$$

b) The angular momentum can be computed using the relation $L_i = I_{ij}\omega_j$, where I_{ij} is the Inertia tensor. In the case at hand,

$$I_{ij} = \text{Diag}\{I, I, (1-a)I\} \tag{2}$$

$$\Rightarrow \vec{L} = I\omega(\sin\alpha\ \hat{n}_1 + (1-a)\cos\alpha\ \hat{n}_3) \tag{3}$$

c) We can compute θ from $\vec{L}\cdot\vec{\omega} = |\vec{L}||\vec{\omega}|\cos\theta$. Substituting for $\vec{L}$ and $\vec{\omega}$ using (1) and (3), we have

$$\theta = \arccos\frac{1 - a\cos^2\alpha}{\sqrt{(1-a)^2\cos^2\alpha + \sin^2\alpha}} \tag{4}$$

d) We are the given the following (useful) information

$$\left(\frac{d\vec{L}}{dt}\right)_s = \left(\frac{d\vec{L}}{dt}\right)_b + \vec{\omega}\times\vec{L} \tag{5}$$

Since there are no external forces on the body, we must have $\left(\frac{d\vec{L}}{dt}\right)_s = 0$. From (5), we can derive the Euler equations of motion (for zero external forces):

$$I_1\dot{\omega}_1 + (I_2 - I_3)\omega_2\omega_3 = 0 \tag{6}$$

$$I_2\dot{\omega}_2 + (I_3 - I_1)\omega_3\omega_1 = 0 \tag{7}$$

$$I_3\dot{\omega}_3 + (I_1 - I_2)\omega_1\omega_2 = 0 \tag{8}$$

where the dotted quantities denote time derivatives and $\vec{\omega} = \omega_1\hat{n}_1 + \omega_2\hat{n}_2 + \omega_3\hat{n}_3$, $I_{ij} = \text{Diag}\{I_1, I_2, I_3\}$.

Since $I_2 = I_3 = I$, (8) yields $\dot{\omega}_3 = 0 \Rightarrow \omega_3 = \omega_3(t=0) = \omega\cos\alpha$. This simplifies (6) and (7) to

$$\begin{aligned} \dot{\omega}_1 + a\omega_3\,\omega_2 &= 0 \quad (9)\\ \dot{\omega}_2 - a\omega_3\,\omega_1 &= 0 \quad (10)\end{aligned}$$

This set of differential equations can be simplified (keeping in mind that ω_3 is constant) by differentiating one and substituting in the other. We obtain

$$\begin{aligned} \ddot{\omega}_1 + (a\omega\cos\alpha)^2\,\omega_1 &= 0 \quad (11)\\ \ddot{\omega}_2 + (a\omega\cos\alpha)^2\,\omega_2 &= 0 \quad (12)\end{aligned}$$

Equations (11) and (12) have oscillatory solutions. In order to completely determine the motion, we need to specify appropriate initial conditions. At $t = 0$, we have $\vec{\omega} = \omega\,(\sin\alpha, 0, \cos\alpha)$. Since (11) and (12) are second order differential equations, we also need to specify the initial values for the first derivative of $\vec{\omega}$. We can determine these by substituting into the Euler equations (6), (7), (8). We have $\dot{\vec{\omega}} = (0, a\omega^2\cos\alpha\sin\alpha, 0)$. Solving the differential equations (11) and (12) yields

$$\vec{\omega} = \omega\,(\sin\alpha\cos(a\omega\cos\alpha\, t), \sin\alpha\sin(a\omega\cos\alpha\ t), \cos\alpha) \quad (13)$$

The angular frequency of rotation about the symmetry axis, is given by the component of $\vec{\omega}$ along $\hat{n}_3$. Therefore,

$$\Omega_b = \omega\cos\alpha \quad (14)$$

e) The precession frequency Ω_p is the angular velocity component along the (conserved) angular momentum $\vec{L}$. Therefore

$$\Omega_p = \frac{\vec{\omega}\cdot\vec{L}}{|\vec{L}|} \quad (15)$$

$\vec{\omega}\cdot\vec{L}$ can be evaluated in terms of the angle α as

$$\vec{\omega}\cdot\vec{L} = I\omega^2(1 - a\cos^2\alpha) \quad (16)$$

We then substitute for α using the expression for $|\vec{L}|$, which is

$$|\vec{L}|^2 = L^2 = I^2\omega^2(\sin^2\alpha + (1-a)^2\cos^2\alpha) \quad (17)$$

Therefore the answer is

$$\Omega_p = \frac{I\omega^2}{(2-a)L}\left(1 - a + \left(\frac{L}{I\omega}\right)^2\right) \quad (18)$$