Fall 2012 EM question 1 correction

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1 Part C

Issue with current solution: In the solutions, position vector in space, labeled $\vec{s}$, and the radial cylindrical coordinate, labeled r are mixed up.

The position vector is:

$$\vec{s} = r\hat{r} + z\hat{z} \tag{1}$$

and thus

$$\begin{aligned} \vec{E} &= e\frac{\vec{s}}{s^3} = e\frac{r\hat{r} + (z+d)\hat{z}}{(r^2 + (z+d)^2)^{3/2}} \\ \vec{B} &= g\frac{\vec{s}}{s^3} = g\frac{r\hat{r} + (z-d)\hat{z}}{(r^2 + (z-d)^2)^{3/2}} \end{aligned} \tag{2}$$

But when calculating the angular momentum density, we should use a position vector in space, **NOT** the radial cylindrical coordinate.

Thus eq(12) **SHOULD** be

$$\begin{aligned} \vec{l} &= \vec{s} \times \frac{\vec{S}}{c^2} \\ &= \frac{1}{4\pi c}\vec{s} \times (\vec{E} \times \vec{B}) \end{aligned} \tag{3}$$

I apologize for the awful notation, $\vec{S}$ is the pointing vector while $\vec{s}$ is our position vector. I wanted to be consistent with the notation of the solutions so, along we chug.

Another issue in the solutions is that $\vec{r}$, which as I argued above should be $\vec{s}$, (magically) becomes a unit vector in eq(14) without pulling out a $\frac{1}{r}$ term, which does not seem to be correct.

The equation should proceed as:

$$\begin{aligned}\vec{l} &= \vec{s} \times \frac{\vec{S}}{c^2} \\ &= \frac{1}{4\pi c} \vec{s} \times (\vec{E} \times \vec{B}) \\ &= \frac{1}{4\pi c} \vec{E}(\vec{s} \cdot \vec{B}(\vec{s}) - \vec{B}(\vec{s} \cdot \vec{E}(\vec{s}))\end{aligned}$$

Note: Though the position vectors inside the $\vec{E}$ and $\vec{B}$ fields are shifted due to the charges' offset from the origin, the position vector we are using in the dot product is just a random vector in R^3 (and thus not shifted) .

$$\begin{aligned}\vec{s} \cdot \vec{B} &= (r\hat{r} + z\hat{z}) \cdot (g\frac{r\hat{r} + (z-d)\hat{z}}{(r^2 + (z-d)^2)^{3/2}}) \\ &= \frac{g}{(r^2 + (z-d)^2)^{3/2}}[r^2 + z(z-d)]\end{aligned}$$

Thus

$$\vec{E}(\vec{s} \cdot \vec{B}) = (\frac{e(r\hat{r} + (z+d)\hat{z})}{(r^2 + (z+d)^2)^{3/2}})(\frac{g(r^2 + z(z-d))}{(r^2 + (z-d)^2)^{3/2}}) \tag{4}$$

similarly

$$\vec{B}(\vec{s} \cdot \vec{E}) = (\frac{g(r\hat{r} + (z-d)\hat{z})}{(r^2 + (z-d)^2)^{3/2}})(\frac{e(r^2 + z(z+d))}{(r^2 + (z+d)^2)^{3/2}}) \tag{5}$$

Next we should subtract (4) from (5) :
They share the same denominator so let's work with just the numerators:

$$(eg[r^3\hat{r} + rz(z-d)\hat{r} + r^2(z+d)\hat{z} + z(z^2 - d^2)\hat{z}]) - eg[r^3\hat{r} + rz(z+d)\hat{r} + r^2(z-d)\hat{z} + z(z^2 - d^2)\hat{z}])$$
$$= eg[-2drz\hat{r} + 2dr^2\hat{z}]$$

Thus:

$$\vec{E}(\vec{s} \cdot \vec{B}) - \vec{B}(\vec{s} \cdot \vec{E}) = \frac{eg[-2drz\hat{r} + 2dr^2\hat{z}]}{(r^2 + (z+d)^2)^{3/2})((r^2 + (z-d)^2)^{3/2})}\hat{z} \tag{6}$$

Now, let's integrate the $\hat{r}$ term over all space

$$\int_0^\infty 2\pi r dr \int_{-\infty}^\infty dz \frac{eg[-2drz]}{(r^2 + (z+d)^2)^{3/2})((r^2 + (z-d)^2)^{3/2})} = 0 \tag{7}$$

We only need to do the dz integral since that is = 0. This is due to the function being an odd function symmetric about 0. Thus we are left with the $\hat{z}$ integral and we can carry on follwoing the original solutions posted.