STATISTICAL MECHANICS

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Statistical Mechanics 1: Magnetic Properties

Consider a *paramagnetic material* with randomly distributed magnetic spin $\boldsymbol{\mu}$ which subjected to the efficient magnetic field $\mathbf{H}_{\text{eff}}$.
The magnetic interaction energy is given by

$$U(\theta) = -\mu_0 H_{\text{eff}} \cos\theta. \tag{1}$$

The probability of a spin $\boldsymbol{\mu}$ which makes with the $\mathbf{H}_{\text{eff}}$ an angle θ is resulted from Boltzmann distribution

$$P_\theta = P_U \frac{dU}{d\theta} = A \exp\left(\frac{\mu_0 H_{\text{eff}} \cos\theta}{k_B T}\right) \mu_0 H_{\text{eff}} \sin\theta. \tag{2}$$

We can find A which is based on normalization condition

$$\int_0^\pi P(\theta) d\theta = 1 \tag{3}$$

$$2Ak_B T \sinh\left(\frac{\mu_0 H_{\text{eff}}}{k_B T}\right) = 1 \tag{4}$$

$$\frac{1}{2k_B T \sinh\left(\frac{\mu_0 H_{\text{eff}}}{k_B T}\right)} = A. \tag{5}$$

The total magnetic moment of the paramagnetic material is

$$M = N \int_0^\pi (\mu_0 \cos\theta) P(\theta) d\theta \tag{6}$$

$$= \frac{N\mu_0^2 H_{\text{eff}}}{2k_B T \sinh\left(\frac{\mu_0 H_{\text{eff}}}{k_B T}\right)} \int_0^\pi \exp\left(\frac{\mu_0 H_{\text{eff}} \cos\theta}{k_B T}\right) \cos\theta \sin\theta d\theta. \tag{7}$$

$$= N\mu_0 \left[\coth\left(\frac{\mu_0 H_{\text{eff}}}{k_B T}\right) - \frac{k_B T}{\mu_0 H_{\text{eff}}}\right] \tag{8}$$

We can use approximation in two cases

$$M \approx \begin{cases} \dfrac{N\mu_0^2 H_{\text{eff}}}{3k_B T} & \text{when } \dfrac{\mu_0 H_{\text{eff}}}{k_B T} \ll 1 \\ N\mu_0 & \text{when } \dfrac{\mu_0 H_{\text{eff}}}{k_B T} \gg 1 \ \ (\textit{Saturated}) \end{cases}. \tag{9}$$

(a) In thermal equilibrium, the spin $\boldsymbol{\mu}$ points both in two directions, $\mathbf{H}_{\text{eff}}$ and $-\mathbf{H}_{\text{eff}}$. We have the discrete probability according to Boltzmann distribution

$$P_{\pm} \propto \exp\left(\mp\frac{\mu_0 H_{\text{eff}}}{k_B T}\right) \tag{10}$$

We have the total magnetic moment

$$\begin{aligned} M &= N\mu_0 \frac{P_- - P_+}{P_- + P_+} && (11)\\ &= N\mu_0 \frac{\exp\left(\frac{\mu_0 H_{\text{eff}}}{k_B T}\right) - \exp\left(-\frac{\mu_0 H_{\text{eff}}}{k_B T}\right)}{\exp\left(\frac{\mu_0 H_{\text{eff}}}{k_B T}\right) + \exp\left(-\frac{\mu_0 H_{\text{eff}}}{k_B T}\right)} && (12)\\ &= N\mu_0 \tanh\left(\frac{\mu_0 H_{\text{eff}}}{k_B T}\right). && (13) \end{aligned}$$

Again, we can use approximation in two cases

$$M \approx \begin{cases} \dfrac{N\mu_0^2 H_{\text{eff}}}{k_B T} & \text{when } \dfrac{\mu_0 H_{\text{eff}}}{k_B T} \ll 1 \\ N\mu_0 & \text{when } \dfrac{\mu_0 H_{\text{eff}}}{k_B T} \gg 1 \ \ (\textit{Saturated}) \end{cases}. \tag{14}$$

(b) We have the equation to solve for M in a self consistent manner

$$M = N\mu_0 \tanh\left(\frac{\mu_0(H_{\text{eff}} + qM))}{k_B T}\right). \tag{15}$$

Take approximation

$$M \approx \begin{cases} \dfrac{N\mu_0^2 H_{\text{eff}}}{k_B T} \dfrac{1}{1 - \dfrac{N\mu_0^2 q}{k_B T}} & \text{when } \dfrac{\mu_0(H_{\text{eff}} + qM)}{k_B T} \ll 1 \\ N\mu_0 & \text{when } \dfrac{\mu_0(H_{\text{eff}} + qM)}{k_B T} \gg 1 \ \ (\textit{Saturated}) \end{cases}. \tag{16}$$

(c) Condition for a ferromagnetic material to have a *spontaneous alignment* is $\begin{cases} H & = 0 \\ M & \neq 0 \end{cases}$.

From (Eq.15) we have

$$\begin{aligned} \frac{M}{N\mu_0} &= \tanh\left(\frac{N\mu_0^2 q}{k_B T}\frac{M}{N\mu_0}\right) && (17)\\ x &= \tanh(\alpha x). && (18) \end{aligned}$$

Here we put $x = \dfrac{M}{N\mu_0}$ and $\alpha = \dfrac{N\mu_0^2 q}{k_B T}$.

To satisfy the condition that $M \neq 0$, it has to be

$$\begin{aligned} \frac{d}{dx}\left(\tanh(\alpha x)\right)\Big|_{x=0} &> \frac{d}{dx}(x). && (19)\\ \alpha &> 1 && (20) \end{aligned}$$

Or we could say $T < T_C = \dfrac{N\mu_0^2 q}{k_B}$.

(d) In the vicinity of T_C, which is $T = T_C - \delta T$ ($\delta T > 0$), we can conclude $x \ll 1$.

$$\alpha = \frac{N\mu_0^2 q}{k_B(T_C - \delta T)} \tag{21}$$

$$= \frac{N\mu_0^2 q}{k_B T_C}\left(1 + \frac{\delta T}{T_C}\right) = 1 + \frac{\delta T}{T_C}. \tag{22}$$

Solve approximately for x we have

$$x \approx \alpha x - \frac{1}{3}\alpha^3 x^2 \tag{23}$$

$$x = \sqrt{3\frac{\alpha - 1}{\alpha^3}} \tag{24}$$

$$x \approx \sqrt{3\frac{\delta T}{T_C}\left(1 - 3\frac{\delta T}{T_C}\right)} \tag{25}$$

$$x \approx \sqrt{3\frac{\delta T}{T_C}} \tag{26}$$

$$\frac{M}{N\mu_0} \approx \sqrt{3}\left(1 - \frac{T}{T_C}\right)^{1/2}. \tag{27}$$

(e) When $T \to T_C$, $x \ll 1$, we have

$$\frac{M}{N\mu_0} \approx \frac{\mu_0 H}{k_B T} + \frac{\mu_0 q M}{k_B T} \tag{28}$$

$$M = \frac{N\mu_0^2 H}{k_B\left(T - \frac{N\mu_0^2 q}{k_B}\right)} = \frac{N\mu_0^2 H}{k_B(T - T_C)}. \tag{29}$$

We have the susceptibility of the material

$$\chi = \lim_{H\to 0}\left(\frac{\partial M}{\partial H}\right) = \frac{N\mu_0^2}{k_B(T - T_C)}. \tag{30}$$

Statistical Mechanics 2: Electron and Helium Gases

(a) Density of states: $g(E) = (2S+1)V\dfrac{(2m_e)^{3/2}}{4\pi^2\hbar^3}E^{1/2}dE$.

Fermi-Dirac distribution: $f(E) = \dfrac{1}{\exp\left(\dfrac{E-\mu}{k_BT}\right)+1}$. At absolute zero, $f(E) = 1$.

Number of electron particles (fermions) is given by

$$\tag{31}$$

As a result, we have the Fermi level energy

$$E_F = \frac{\hbar^2}{2m_e}(3\pi^2 n_1)^{3/2}. \tag{32}$$

We have the pressure of fermions gas

$$p_e = \frac{2}{3}\frac{U}{V} = \frac{4}{3}\frac{(2m_e)^{3/2}}{4\pi^2\hbar^3}\int_0^{E_F} E^{3/2}dE \tag{33}$$

$$= \frac{\hbar^2}{5m_e}(3\pi^2)^{2/3}n_1^{5/3}. \tag{34}$$

We have the relation

$$p_e = p_{He} \tag{35}$$

$$\frac{\hbar^2}{5m_e}\left(3\pi^2\right)^{2/3}n_1^{5/3} = n_2 k_B T. \tag{36}$$

(b) We have Sommerfeld expansion for the chemical potential

$$\mu(T) \approx E_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_BT}{E_F}\right)^2 + \dots\right] \tag{37}$$

To satisfy the condition that the electron gas is effectively at absolute zero

$$\frac{k_BT}{E_F} \ll \frac{2\sqrt{3}}{\pi} \tag{38}$$

$$n_2 k_B T = \frac{2}{5}n_1 E_F \ll \frac{2\sqrt{3}}{\pi}n_2 E_F \tag{39}$$

$$\frac{n_2}{n_1} \gg \frac{\pi}{5\sqrt{3}} \approx 0.36. \tag{40}$$

(c) Bose-Einstein distribution: $f(E) = \dfrac{1}{\exp\left(\dfrac{E-\mu}{k_BT}\right)-1}$.

For a Boltzmann gas, $\dfrac{E-\mu}{k_BT} \gg 1$ and $\exp\left(\dfrac{\mu}{k_BT}\right) \ll 1$.

Number of helium particles (bosons) is given by

$$N = \frac{(2m_{\rm He})^{3/2}V}{4\pi^2\hbar^3}\int_0^\infty \frac{E^{1/2}dE}{\exp\left(\frac{E-\mu}{k_BT}\right)-1} \tag{41}$$

$$\approx \exp\left(\frac{\mu}{k_BT}\right)\frac{(2m_{\rm He})^{3/2}V}{4\pi^2\hbar^3}\int_0^\infty E^{1/2}\exp\left(-\frac{E}{k_BT}\right)dE \tag{42}$$

$$\approx \exp\left(\frac{\mu}{k_BT}\right)\frac{(2m_{\rm He})^{3/2}V}{4\pi^2\hbar^3}(k_BT)^{3/2}\int_0^\infty x^{1/2}e^{-x}dx \tag{43}$$

$$= \exp\left(\frac{\mu}{k_BT}\right)\left(\frac{2m_{\rm He}k_BT}{\pi}\right)^{3/2}\frac{V}{8\hbar^3} \tag{44}$$

$$n_2 \ll \left(\frac{2m_{\rm He}k_BT}{\pi}\right)^{3/2}\frac{V}{8\hbar^3}. \tag{45}$$

From (Eqs. 36 and 45) we can solve

$$\frac{n_2}{n_1} \ll \left[\frac{72\pi}{125}\left(\frac{m_p}{m_e}\right)^3\right]^{1/5} \approx 102.26. \tag{46}$$

(d) Base on least error analysis, we can conclude the number that satisfy both conditions is

$$\frac{n_2}{n_1} = \left[\prod_{i=1}^{k}\left(\frac{n_2}{n_1}\right)_i\right]^{1/k} \approx 6. \tag{47}$$

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