EM solutions, Fall 2021 Written Exam

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Problem 1.

a) We will expand the (azimuthally symmetry) potential in terms of Legendre polynomials

$$\Phi(r > R, \theta) = \sum_{l=0}^{\infty} B_l r^{-l+1} P_l(\cos\theta)$$

as these are the solutions to Laplace's equation. Now there are also solutions of the form $r^l P_l$, but they are not well behaved at spatial ∞, so we discard them. Using orthogonality, we see that

$$B_l = \frac{2l+1}{2R^{-(l+1)}} \int_{-1}^{1} d\cos\theta \; \Phi(R,\theta) P_l(\cos\theta)$$

Now as the hemispheres are at potential $\pm V$ for $r = R$, we can substitute in for Φ above to find

$$B_l = \frac{2l+1}{2R^{-(l+1)}} V(1 - (-1)^l) \int_{0}^{1} d\cos\theta \; P_l(\cos\theta)$$

We see that only the Legendre polynomials of odd degree remain. Keeping the first two nonvanishing B_l and substituting in for P_l, we can write

$$\Phi(r,\theta) \sim V \left(\frac{3}{2} \left(\frac{R}{r}\right)^2 \cos\theta - \frac{7}{8} \left(\frac{R}{r}\right)^4 \frac{1}{2}(5\cos^3\theta - 3\cos\theta) + \cdots \right)$$

b) In the near zone, long wavelength regime, we can use the dipole approximation, so we only keep the first term in the potential. As the long wavelength regime is also the low frequency limit, we can also take a quasistatic approximation where we simply substitute in $V \to V\cos\omega t$. Therefore, we have

$$\Phi(r,\theta) \approx \frac{3}{2} \frac{R^2}{r^2} V \cos\omega t \cos\theta$$

Note this is of the suggested dipole form with dipole moment $\vec{p}(t) = 6\pi\epsilon_0 V R^2 \cos\omega t \hat{z}$.

c) Following the formulas given to us and using that $\hat{n} \times \hat{z} = -\sin\theta\hat{\phi}$, we calculate that

$$\vec{H} = -\frac{3}{2} \frac{\epsilon_0 c k^2 e^{ikr}}{r} V \cos(\omega t) R^2 \sin\theta\hat{\phi}, \quad \vec{E} = \frac{3}{2} \frac{k^2 e^{ikr}}{r} V \cos(\omega t) R^2 \sin\theta\hat{\theta}$$

The angular distribution of radiated power is given by calculating the Poynting vector so that

$$\frac{dP}{d\Omega} = r^2 \vec{S} \cdot \hat{n} = \frac{9}{8} c\epsilon_0 V^2 \sin^2\theta k^4 R^2$$

Note in the above expression, we have added in an extra factor of $1/2$ to account for time averaging over the $\cos^2 \omega t$ term. The total radiated power is

$$P_{tot} = \frac{9}{8} V^2 c\epsilon_0 k^4 R^4 \int d\Omega \ \sin^2\theta = 3\pi V^2 c\epsilon_0 k^4 R^4$$

where we have used $\int d\Omega \ \sin^2\theta = 2\pi \int_0^\pi d\theta \ \sin^3\theta = 2\pi \int d\theta \ (\sin\theta - \sin\theta\cos^2\theta) = 2\pi(-\cos\theta + \cos^3\theta/3)\big|_0^\pi = 8\pi/3$.

d) This antenna does not radiate as an electric quadrupole. This is because from the first part, we saw that it does not have a quadrupole term in the expansion of the potential $\Phi(r,\theta)$. Such a term would scale as $r^{-3}P_2(\cos\theta)$. Instead, its next nonvanishing multipole moment that will radiate is the octupole moment.

Problem 2.

a) Such an infinitesimal displacement leads to a charge density change $\delta\rho_e(\vec{s}) = \rho_e(\vec{s} - \delta\vec{s}) - \rho_e(\vec{s}) = -\nabla\rho_e(\vec{s}) \cdot \delta\vec{s}$, so that

$$\delta V = \int d^3s\ (\delta\rho_e(\vec{s}))\varphi_N(\vec{r} - \vec{s}) = -\int d^3s\ \varphi_N(\vec{r} - \vec{s})\nabla\rho_e(\vec{s}) \cdot \delta\vec{s}$$

Integrating by parts leads us to

$$\delta V = \int d^3s\ \rho_e(\vec{s})\vec{E}_N(\vec{r} - \vec{s}) \cdot \delta\vec{s}$$

which leads to $\vec{F} = \int d^3s\rho_e(\vec{s})\vec{E}_N(\vec{r} - \vec{s})$.[1]

b) Using the Poisson equation and integrating by parts twice (neglecting boundary terms), we obtain

$$V_E(\vec{r}) = \int d^3s\ \rho_N(\vec{r} - \vec{s})\varphi_e(\vec{s})$$

c) Using that the neutron charge density is spherically symmetric, we can relabel

$$V_E(\vec{r}) = \int d^3s\ \rho_N(-\vec{r} + \vec{s})\varphi_e(\vec{s}) = \int d^3s\ \rho_N(\vec{s})\varphi_e(\vec{r} + \vec{s})$$

Now taylor expanding φ_e, we have (summing over repeated Cartesian indices)

$$V_E(\vec{r}) = \left(\int d^3s\ \rho_N(\vec{s})\right)\varphi_e(\vec{r}) + \left(\int d^3s\ \rho_N(\vec{s})s_i\right)\partial_i\varphi_e(\vec{r}) + \frac{1}{2}\left(\int d^3s\ \rho_N(\vec{s})s_is_j\right)\partial_i\partial_j\varphi_e(\vec{r}) + \cdots$$

The first term vanishes as the neutron has zero total charge, while the second term vanishes because ρ_N is even while s_i is odd. Spherical symmetry means only $i = j$ terms survive in the third term, so

$$V_E(\vec{r}) = \frac{1}{2}\partial_i^2\varphi_e(\vec{r})\int d^3s\ s_i^2\rho_N(\vec{s}) = \frac{1}{6}\nabla^2\varphi_e(\vec{r})\int d^3s\ s^2\rho_N(\vec{s})$$

where we have used $\int d^3s\ s_i^2\rho_N = \int d^3s\ s^2\rho_N/3$ by spherical symmetry. Using that $\nabla^2\varphi_e(\vec{r}) = -\frac{e}{\epsilon_0}\delta^3(\vec{r} - \vec{s}_0)$, we then have

$$V_E(\vec{r}) = -\frac{1}{6}\frac{e}{\epsilon_0}\delta^3(\vec{r} - \vec{s}_0)\int d^3s\ s^2\rho_N(\vec{s})$$

from which we can write the scattering amplitude as

$$\int d^3r\ V_E(\vec{r}) = -\frac{1}{6}\frac{e}{\epsilon_0}\int d^3s\ s^2\rho_N(\vec{s})$$

d) A measurement of the low energy cross section reveals the effective/rms charge radius of the neutron (aka the second moment of ρ_N).

[1] Note this is a little different compared to the problem statement, but I am not sure if the problem statement is incorrect or if I made a mistake (or maybe something wrong with the notation for $\vec{s}$?). Using my method above, the problem statement result comes from initially writing $V = \int d^3s\ \rho_e(\vec{s})\varphi_N(\vec{s})$.