

SM solutions, Fall 2021 Written Exam

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Problem 1.

a) Filling all the wavevectors below $k_{F\pm}$ until the desired number densities are reached, we must solve for

$$n_{\pm} = \frac{N_{\pm}}{V} = \int \frac{d^3k}{(2\pi)^3} \Theta(k_{F\pm} - k) = \int_0^{k_{F\pm}} \frac{4\pi k^2 dk}{(2\pi)^3}$$

which leads to

$$k_{F\pm} = (6\pi^2 n_{\pm})^{1/3}$$

b) From the quadratic dispersion $E_{kinetic} = \hbar^2 k^2 / 2m$, we can find the kinetic energy density at zero temperature to be

$$\epsilon_{\pm,kin} = \int_0^{k_{F\pm}} \frac{d^3k}{(2\pi)^3} \hbar^2 k^2 / 2m = \int_0^{k_{F\pm}} \frac{\hbar^2 k^4 dk}{4m\pi^2} = \frac{\hbar^2 k_{F\pm}^5}{20m\pi^2}$$

Substituting in for the Fermi wavevectors from last part, we find

$$\epsilon_{tot,kin} = \epsilon_{+,kin} + \epsilon_{-,kin} = \frac{\hbar^2}{20m\pi^2} (k_{F+}^5 + k_{F-}^5) = \frac{3\hbar^2}{10m} (6\pi^2)^{2/3} (n_+^{5/3} + n_-^{5/3})$$

c) We expand the above result using the binomial series expansion

$$(x + \Delta)^{5/3} \sim x^{5/3} + \frac{5}{3} x^{2/3} \Delta + \frac{10}{9} \frac{1}{2} x^{-1/3} \Delta^2 - \frac{10}{27} \frac{1}{6} x^{-4/3} \Delta^3 + \frac{40}{81} \frac{1}{24} x^{-7/3} \Delta^4 - \dots$$

to find

$$\epsilon_{tot,kin} = \frac{3\hbar^2}{10m} (6\pi^2)^{2/3} \left(\left(\frac{n}{2}\right)^{5/3} + \frac{5}{3} \left(\frac{n}{2}\right)^{5/3} \delta + \frac{10}{9} \frac{1}{2} \left(\frac{n}{2}\right)^{5/3} \delta^2 - \frac{10}{27} \frac{1}{6} \left(\frac{n}{2}\right)^{5/3} \delta^3 + \frac{40}{81} \frac{1}{24} \left(\frac{n}{2}\right)^{5/3} \delta^4 + \dots \right) + (\delta \rightarrow -\delta)$$

The terms odd in δ cancel out and we are left with

$$\epsilon_{tot,kin} = \frac{3\hbar^2}{5m} (6\pi^2)^{2/3} \left(\frac{n}{2}\right)^{5/3} \left[1 + \frac{5}{9} \delta^2 + \frac{5}{243} \delta^4 + \dots \right]$$

d) The interaction density is of the form

$$\epsilon_{int} = \frac{U}{V} = \alpha n_+ n_- = \alpha \frac{n^2}{4} (1 - \delta^2)$$

The total energy (density) can then be written as a series in δ

$$\epsilon_{tot} = \epsilon_{int} + \epsilon_{tot,kin} = f_0(n) + f_2(n)\delta^2 + f_4(n)\delta^4 + \dots$$

where the $f_i(n)$ are from the series expansion from last part and the interaction term we just calculated. The Stoner instability occurs when the electron gas

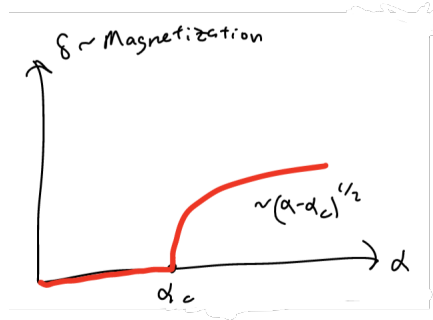
can decrease its total energy for small, nonzero δ . This requires that $f_2(n) < 0$. As we can calculate

$$f_2(n) = \frac{\hbar^2}{3m} (6\pi^2)^{2/3} \left(\frac{n}{2}\right)^{5/3} - \alpha \frac{n^2}{4}$$

we see that

$$f_2(n) < 0 \implies \alpha > \alpha_c = \frac{2\hbar^2}{3m} (3\pi^2)^{2/3} n^{-1/3}$$

d) For $\alpha < \alpha_c$, the minimum energy is given by $\delta = 0$. A continuous phase transition occurs at $\alpha = \alpha_c$, which we can see because $f_4(n) > 0$, so that $\delta \rightarrow 0$ continuously as $\alpha \rightarrow \alpha_c$ from above. For small δ , the shape of δ scales as $(\alpha - \alpha_c)^{1/2}$, which can be seen by minimizing the energy $\partial_\delta \epsilon_{tot} = 0$. Therefore, the magnetization will also scale as $(\alpha - \alpha_c)^{1/2}$ as the magnetization is directly proportional to δ .



Problem 2.

a) The critical point can be obtained through setting dP/dn and d^2P/dn^2 to zero, as these conditions locate the transition between the isotherms in which P is an increasing/decreasing function of n .

To justify the above construction, we note that an instability will occur when an increase in volume leads to an increase in pressure. In other words, whenever δV and δP have the same sign. Since $\delta n = -N\delta V/V^2$, we see that the equation of state will be invalid if the pressure is a decreasing function of the density n , as then

$$\delta P \delta V \propto (\delta V)^2 > 0$$

and an instability will occur. Now starting from the equation of state, we obtain

$$\frac{d^2P}{dn} \implies n_c = \frac{b}{c}$$

from which

$$\frac{dP}{dn} = 0 \implies k_B T_c = \frac{b^2}{2c}$$

Lastly, we can find $P_c = \frac{b^3}{6c^2}$ and $k_B T_c n_c / P_c = 3$.

b) We first rewrite

$$\kappa_T = -\frac{1}{V} \partial_P V \Big|_T = \frac{1}{n} \partial_n P \Big|_T$$

from which we obtain

$$\kappa_T = \frac{1}{n} \frac{1}{k_B T - bn + cn^2/2} \propto \frac{1}{T - T_c}$$

at $n = n_c$. Therefore, we see the isothermal compressibility diverges near T_c .

c) On the critical isotherm $T = T_c$,

$$P - P_c = \frac{b^2}{2c}n - \frac{b}{2}n^2 + \frac{c}{6}n^3 - \frac{b^3}{6c^2} = \frac{c}{6} \left(n^3 - 3\frac{b}{c}n^2 - 3\frac{b^2}{c^2}n - \frac{b^3}{c^3} \right) = \frac{c}{6}(n - n_c)^3$$

d) We use the hint that $d\mu = dP/n$, which can be derived from the Gibbs-Duhem relation. When the system is phase separated, the chemical potentials for the liquid and gas will be the same. Integrating $d\mu = dP/n$ leads to the Maxwell construction

$$0 = \int_{n_-}^{n_+} \frac{dP}{n} = \int_{n_c(1-\delta)}^{n_c(1+\delta)} \frac{dn}{n} (k_B T - bn + cn^2/2)$$

Now substituting in for n_c , we find

$$0 = k_B T \ln \frac{1 + \delta}{1 - \delta} - \frac{b^2}{c} \delta$$

and substituting in for T_c , we find that δ is given implicitly by

$$\delta = \frac{1}{2} \frac{T}{T_c} \ln \frac{1 + \delta}{1 - \delta}$$

Series expanding the logarithm, we find

$$\delta \sim T^{-1/2} (T - T_c)^{1/2}$$

as $(T_c - T) \rightarrow 0$.